HISTORY OF MATHEMATICS IN INDIA AND OUR SCHOOL MATHEMATICS CURRICULUM

M. D. SRINIVAS
Centre for Policy Studies, Chennai
mdsrinivas50@gmail.com

OUTINE

- Introduction: Most of the mathematics taught in our schools was developed and widely taught in India for over 1500 years. These and other Indian contributions in Mathematics find no place in our school curricula.
- A brief overview of development of mathematics in India.
- Suggestions for incorporating Indian mathematical contributions in our school mathematics curriculum.

INTRODUCTION

- In recent times, there are many scholarly reviews and histories of mathematics, which discuss the seminal contributions of Indian mathematicians and indicate how these contributions have influenced the development of mathematics the world over. Several studies have also highlighted the unique algorithmic and constructive approach of Indian mathematics, which is indeed known as *Gaṇita* or the science of computation.

- It is now fairly established that much of the mathematics, which is currently taught at the school level, largely originated in India and formed a major component of the popular education, all over the country, for more than a millennia and a half. While the great mathematical classics such *Līlāvatī* and *Bījagaṇita* of Bhāskarācārya (1150 CE) served as reputed text-books at a pan-Indian level, many popular texts continued to be written in Sanskrit, as well as the regional languages of India, till two centuries ago.

- Mathematics happens to be a core subject that is learnt by over 30 crore students in India–comprising all the students in classes I-X, and a large fraction of the students in classes XI- XII. Unfortunately the curriculum and text-books of mathematics, which have been followed in our country make no mention, whatsoever, of Indian contributions to mathematics.

INTRODUCTION

- While a miniscule fraction of our students may be aware of the names of great mathematicians such as Āryabhaṭa, Brhmagupta, Bhāskarācārya and Mādhava, few among them would be able to recount any of the significant contributions made by these or other famous Indian mathematicians. The brief historical notes presented in some of our mathematics textbooks (for instance, the current text-books of Grades XI and XII) are devoted to the contributions made by the Greeks in the ancient period, or by the Europeans in the modern period.
- It is preposterous that the large body of mathematics that originated and developed in India is taught to our students as if it has got nothing to do with Indian culture and history. This is further aggravated by the choice of topics that are included in the curriculum which follows the pattern set during the period of colonial rule, and ignores many of the well-known Indian contributions which could be profitably taught to students at the school level–such as the solution of indeterminate equations, combinatorial techniques, construction of magic squares, properties of cyclic quadrilaterals, computation of π as well as the sine and cosine functions, etc.
- Introduction of such topics in our school curriculum would enable our students to develop greater interest in the subject, and an appreciation of the Indian contributions to mathematics. Introduction of such topics would also serve to highlight the algorithmic approach of *Gaṇita* or Indian Mathematics, and would be a crucial step in any endeavour to rekindle the Indian genius for mathematics.

DEVELOPMENT OF INDIAN ASTRONOMY & MATHEMATICS

Ancient & Early Classical Period (Prior to 500 BCE)

- Astronomical Observations recorded in the Vedic corpus: Luni-Solar year, intercalation, 27 Nakṣatras, 18 year cycle of eclipses, Equinoxes and Solstices at different epochs, Abhaya-Dhruva, the Pole Star at 2800 BCE.
- Mathematical knowledge as revealed in the Vedic Corpus. Decimal place-value nomenclature for numbers. Geometry of Vedis (alters).
- Knowledge of Astronomy and Mathematics as revealed in ancient archaeological sites.
- *Vedāṅgajyotiṣa* (c. 1350 – 1150 BCE): Computation of the motion of Sun and the Moon based on a five year Yuga cycle.
- *Parāśaratantra* (c.1350 - 1150 BCE): Seasons defined in terms of the Nakṣatra division of Zodiac, List of ancient Comets, Planetary periods, Visibility of Agastya.
- *Śulvasūtras* (prior to 800 BCE): The oldest texts of geometry. Present procedures for construction and transformation of geometrical figures and alters (Vedi) using rope (Rajju) and gnomon (Śaṅku).

Baudhāyana-Śulvasūtra (Prior to 800 BCE)

- Units of measurement (*Bhūmiparimāṇa*)
- Marking directions and construction of a square of a given side (*Samacaturaśra-karaṇa*)
- Construction of a rectangle and isosceles trapezium of given sides
- Construction of $\sqrt{2}$ (*Dvikaraṇī*), $\sqrt{3}$ and $\left(\frac{1}{\sqrt{3}}\right)$ times a given length
- The square of the diagonal of a rectangle is the sum of the squares of its sides (*Bhujā-Koṭi-Karṇa-Nyāya* – Oldest Theorem in Geometry)

दीर्घचतुरश्रस्याक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी च यत् पृथग्भूते कुरुतस्तदुभयं करोति।

Baudhāyana-Śulvasūtra

- Construction of squares which are the sum and difference of two squares
- Transforming a square into a rectangle, isosceles trapezium, isosceles triangle and a rhombus of equal area and vice versa
- Approximate conversion of a square of side **a** into a circle of radius

$$r \approx \left(\frac{a}{3}\right)(2+\sqrt{2}). \quad [\pi \approx \mathbf{3.0883}]$$

- An approximation for $(2)^{\frac{1}{2}}$ (*dvikaraṇī*):

$$\sqrt{\mathbf{2}} \approx \mathbf{1} + \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{1}}{\mathbf{3.4}} - \frac{\mathbf{1}}{\mathbf{3.4.34}} = \mathbf{1.4142156}$$

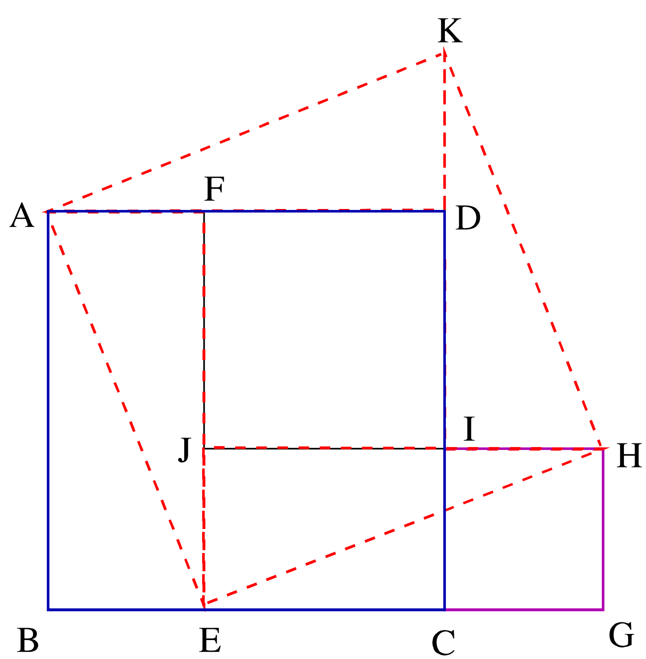
- Positions, relative distances and areas of altars. Shapes of different altars and their construction.

Constructing a square that is sum of unequal squares

An application of the *Śulba*-theorem

नानाचतुरश्रे समस्यन् कनीयसः करण्या वर्षीयसो वृध्रमुल्लिखेत्। वृध्रस्य अक्ष्णयारज्जुः समस्यतोः पार्श्वमानी भवति। *(BSS I.50)*

Desirous of combining different squares, may you mark the rectangular portion of the larger [square] with a side (*karaṇyā*) of the smaller one (*kanīyasaḥ*). The diagonal of this rectangle (*vṛddhra*) is the side of the sum of the two [squares].



- The term *vṛdhra* in the above *sūtra* refers to the rectangle ABEF.
- Asking us to mark this rectangle, all that the text says is the cord AE *akṣṇayārajjuḥ* gives the side of the sum of the squares.
- In other words,

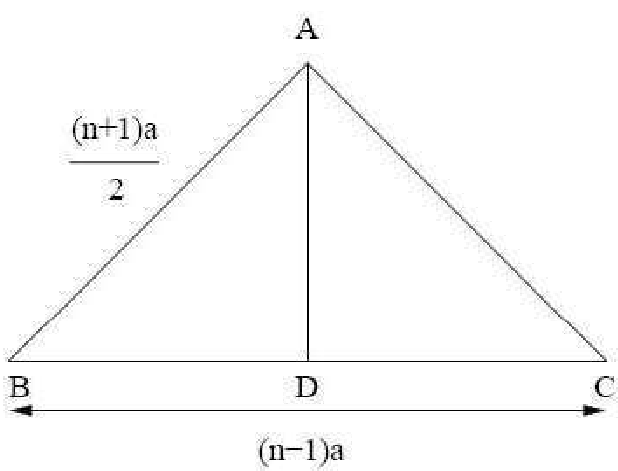
$$\begin{aligned} AE^2 &= ABCD + CGHI \\ &= AB^2 + CG^2 \\ &= AB^2 + BE^2. \end{aligned}$$

Kātyāyana-Śulvasūtra

To construct a square which is *n*-times a given square

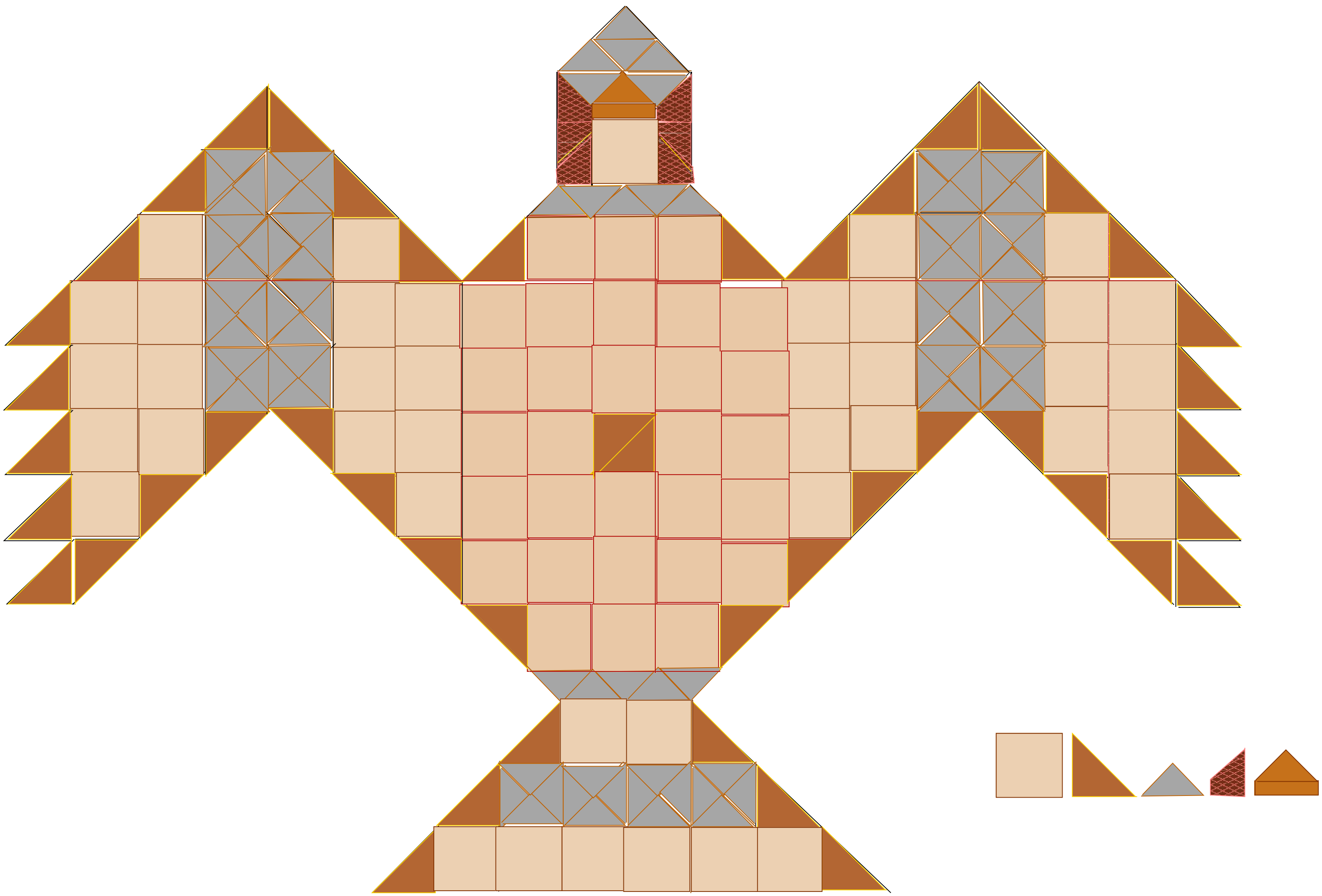
यावत्प्रमाणानि समचतुरश्राण्येकीकर्तुं चिकिर्षेत् एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधिकानि। त्र्यस्त्रिर्भवति तस्येषुस्तत्करोति। (कात्यायनशुल्बसूत्रम् ६.७)

As many squares as you wish to combine into one, the transverse line will be one less than that. Twice the side will be one more than that. That will be the triangle. Its arrow (altitude) will produce that.



$$
\begin{aligned}
AD^2 &= AB^2 - BD^2 \\
&= \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2 \\
&= na^2.
\end{aligned}
$$

Śyenaciti: Falcon-shape



DEVELOPMENT OF INDIAN ASTRONOMY & MATHEMATICS

Classical Period I (500 BCE – 600 CE)

- The Techniques and methodology of Pāṇini's *Aṣṭādhyāyī* (500 BCE).
- *Vṛddhagargasaṁhitā*
- Piṅgala's *Chandaḥsūtra* (prior to c.300 BCE) and the development of binary representation and combinatorics.
- Astronomy and Mathematics in Bauddha and Jaina Texts.
- The emergence of decimal place value system with zero.
- *Āryabhaṭīya* (c.499 CE): The standard procedures in arithmetic, algebra, geometry and trigonometry are perfected by this time. The basic models of planetary motion, computation of eclipses etc., are clearly formulated.
- Works of Varāhamihira (c. 505-587): *Pañcasiddhāntikā*, *Bṛhatsaṁhitā.*
- *Candravākyas* of Vararuci
- Bakṣālī Manuscript.

Piṅgala's Algorithm for Computing Saṅkhyā (2^n)

The number of metres of n-syllables is $S_n = 2^n$.

Piṅgala gives an optimal algorithm for finding 2^n (in fact the n-th power, a^n, of any number a) by means of a much smaller number of multiplication and squaring operations. This algorithm was used for summing geometric series in the later mathematical texts.

द्विरर्धे। रूपे शून्यम्। द्विःशून्ये। तावदर्धे तद्गुणितम्। (छन्दःशास्त्रम् ८.२८-३१)

Halve the number and mark "2". If the number cannot be halved, deduct one and mark "0".

[Proceed till you reach zero. Then, start with number 1 and scan this sequence of marks from the right]

If "0", then multiply by 2. If "2", then square.

Piṅgala's algorithm is based on the relations: $a^{2m} = (a^m)^2$ and $a^{2m+1} = a\,(a^{2m})$

Piṅgala's Algorithm for Computing Saṅkhyā (2^n)

- Halve the number and mark "2".
- If the number cannot be halved deduct one and mark "0".
- Proceed till you reach zero.
- Then, start with number 1 and scan this sequence of marks from the right.
- If "0", then multiply by 2.
- If "2", then square.
- **For n = 21, Piṅgala's algorithm involves only four squarings and two multiplications.**

Example: **Twenty-one-syllable metres (n = 21)**

21 cannot be halved. (21-1) = 20. Deduct 1 and mark "0"

20/2 = 10. Mark "2"

10/2 = 5. Mark "2"

5 cannot be halved. (5-1) = 4. Deduct 1 and Mark "0"

4/2 = 2. Mark "2".

2/2 = 1. Mark "2"

1-1 = 0. Deduct 1 and mark "0"

The sequence of marks: 0, 2, 2, 0, 2, 2, 0

$1\text{x}2$, $(1\text{x}2)^2$, $((1\text{x}2)^2)^2$, $((1\text{x}2)^2)^2\text{x}2$, $(((1\text{x}2)^2)^2\text{x}2)^2$, $((((1\text{x}2)^2)^2\text{x}2)^2)^2$,

$\mathbf{((((1\text{x}2)^2)^2\text{x}2\,)^2)^2\text{x}2 = 2^{21}}$.

$\mathbf{((((a)^2)^2 a)^2\,)^2\text{x}a = a^{21}}$

Varṇa-Meru of Piṅgala

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

The number of metrical forms with r *gurus* (or *laghus*) in the *prastāra* of metres of n-syllables is the binomial coefficient ${}^nC_r$.

Halāyudha's commentary (c.950) on *Piṅgala-sūtras* (c.300 BCE) explains the basic rule for the construction of the above table, which is the recurrence relation

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

DEVELOPMENT OF DECIMAL PLACE VALUE SYSTEM

- The Yajurveda Saṃhitā talks of powers of 10 up to 10^{18} (parārdha)
- The Upaniṣads talk of zero (śūnya, kha) and infinity (pūrṇa).
- Pāṇini's *Aṣṭādhyāyī* uses the idea of zero-morpheme (lopa)
- The Bauddha and Naiyāyika philosophers discuss the notions of śūnya and abhāva.
- Piṅgala's *Chandaḥśāastra* uses zero as a marker (rupe śūnyam)
- Philosophical works such as the works of Vasumitra (c.50 CE) and the Vyāsabhāṣya on Yogasūtra (3.1.3) refer to the way the same symbol acquires different connotations in the place value system.

  यथैका रेखा शतस्थाने शतं दशस्थाने दश एका च एकस्थाने यथा चैकत्वेपि स्त्री माता चोच्यते दुहिता च स्वशचेति।
- The *Āryabhaṭīya* of Āryabhaṭa (c.499 CE) presents all the standard methods of calculation based on the place value system.
- The *Brāshmasphuṭasiddhānta* (c.628 CE) of Brahmagupta gives all the rules of arithmetic of zero and negative numbers.

GAṆITA: INDIAN MATHEMATICS OF COMPUTATION

गण्यते संख्यायते तद् गणितम्। तत्प्रतिपादकत्वेन तत्संज्ञं शास्त्रमुच्यते।

As noted by Gaṇeśa Daivajña, in his commentary *Buddhivilāsinī* (c.1540) on *Līlāvatī* (c.1150), Gaṇita (Indian Mathematics) is the science (art) of computation. Indian Mathematical Texts give rules to describe systematic and efficient procedures of calculation.

Here is an ancient rule for squaring as cited by Bhāskara I (c.629 AD),

अन्त्यपदस्य वर्गं कृत्वा द्विगुणं तदेव चान्त्यपदम्। शेषपदैराहन्यात् उत्सार्योत्सार्य वर्गविधौ॥

In the process for calculating the square, the square of the last digit is found (and placed over it). The rest of the digits are multiplied by twice the last digit (and the results placed over them). Then (omitting the last digit), moving the rest by one place each, the process is repeated again and again.

GAṆITA: INDIAN MATHEMATICS OF COMPUTATION

An Example: To calculate 125^2

$$\begin{array}{rrrrrl}
1 & 5 & 6 & 2 & 5 & \\ \cline{1-5}
 & & & & 25 & 5^2 = 25 \\
 & & 4 & 20 & & 2^2 = 4,\ 5.2.2 = 20 \\
1 & 4 & 10 & & & 1^2 = 1,\ 2.2.1 = 4,\ 5.2.1 = 10 \\ \cline{1-3}
1 & 2 & 5 & & &
\end{array}$$

Note: Even this "ancient" rule of squaring uses $n(n+1)/2$ multiplications for squaring an n-digit number.

The modern word **algorithm** derives from the medieval word `algorism' which referred to the Indian methods of calculation based on the place value system. The word `algorism' itself is a corruption of the name of the Central Asian mathematician al Khwarizmi (c.825) whose *Hisab al Hindi* (Indian Method of Calculation) was the source from which the Indian methods of calculation reached the Western world.

Gaṇitapāda of *Āryabhaṭīya* (499 CE)

The following topics are dealt with in 33 verses of *Gaṇitapāda* of *Āryabhaṭīya*:

- *Saṃkhyāsthāna*: Place values.
- *Vargaparikarma, ghanaparikarma*: Squaring and cubing.
- *Vargamūlānayana*: Obtaining the square-root.
- *Ghanamūlānayana*: Obtaining the cube-root.
- Area of a triangle and volume of an equilateral tetrahedron.
- Obtaining the area of a circle, volume of a sphere.
- Obtaining the area of a trapezium.
- Chord of a sixth of the circumference.
- Approximate value of the circumference ($\pi \approx 3.1416$)

Gaṇitapāda of *Āryabhaṭīya*

- *Jyānayana*: Computing table of Rsines
- *Chāyā-karma*: Obtaining shadows of gnomons.
- *Karṇānayana*: Square of the hypotenuse is the sum of the squares of the sides.
- *Śarānayana*: Arrows of intercepted arcs
- *Śreḍhī-gaṇita*: Summing an AP, finding the number of terms, repeated summations
- *Varga-ghana-saṅkalanānayana*: Obtaining the sum of squares and cubes of natural numbers.
- *Mūlaphalānayana*: Interest and principal
- *Trairāśika*: Rule of three

Gaṇitapāda of *Āryabhaṭīya*

- *Bhinna-parikarma*: Arithmetic of fractions.
- *Pratiloma-karaṇa*: Inverse processes
- *Samakaraṇa-uddeśaka-pradarśana*: Linear equation with one unknown
- *Yogakālānayana*: Meeting time of two bodies
- *Kuṭṭākāra-gaṇita*: Solution of linear indeterminate equation

Thus, by the time of *Āryabhaṭīya*, Indian mathematicians had systematised most of the basic procedures of arithmetic, algebra, geometry and trigonometry that are generally taught in schools to-day, and many more that are more advanced (such as *kuṭṭaka* and sine-tables) and are of importance in astronomy.

Computation of Sines From Second Order Sine-Differences

Computation of Rsine-table (accurate to minutes in a circle of circumference 21,600 minutes), by the method of second-order Rsine-differences, in *Āryabhaṭīya* of Āryabhaṭa (c.499)

प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥

$$
\begin{aligned}
B_j &= Rsin(jh), j = 1, 2, \ldots, 24, h = 225' \\
\triangle_j &= B_{j+1} - B_j
\end{aligned}
$$

Rsines are to be computed from the relations:

$$
\begin{aligned}
\triangle_{j+1} - \triangle_j &= -B_j \left[\frac{(\triangle_1 - \triangle_2)}{B_1} \right] \\
&\approx \frac{-B_j}{B_1} \\
B_1 &\approx 225'
\end{aligned}
$$

Āryabhaṭa's Sine Table

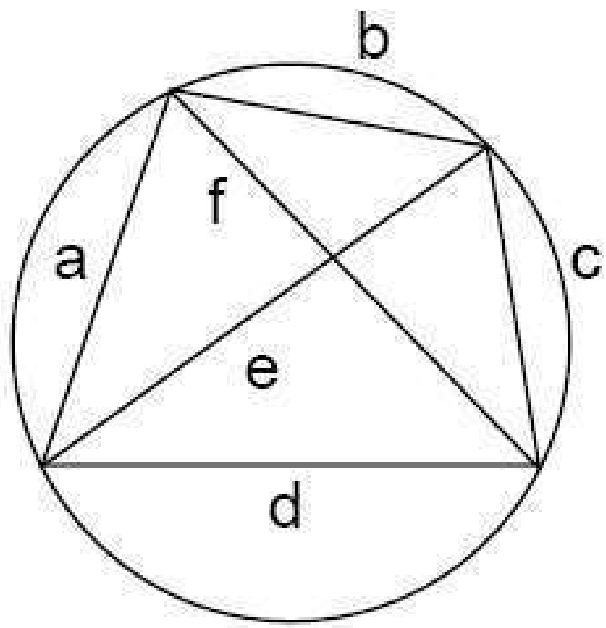
θ in min.	R $\sin\theta$ according to *Āryabhaṭīya*	Govindasvāmi	Mādhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 7
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

DEVELOPMENT OF INDIAN ASTRONOMY & MATHEMATICS

Classical Period II (600 CE – 1200 CE)

- Works of Bhāskara I (c.629): *Āryabhaṭīyabhāṣya*, *Mahābhāskarīya, Laghubhāskarīya*.
- Works of Brahmagupta: *Brāhmasphuṭasiddhānta* (c.628 CE) *Khaṇḍa-khādyaka* (c.665): Mathematics of zero and negative numbers. Development of algebra. Second order interpolation.
- Parahita system of Haridatta (c. 683)
- *Pāṭīgaṇita* of Śrīdhara
- Works of Lalla (c.750), Govindasvāmin (c.800)
- *Gaṇitasārasaṅgraha* of Mahāvīrācārya (c.850)
- Works of Pṛthūdakasvāmin (c.860), Vaṭeśvara (c. 904), Muñjāla (c.932), Āryabhaṭa II (c.950), Śrīpati (c.1039) and Jayadeva (c.1050)
- Works of Bhāskarācārya II (c.1150): *Līlāvatī, Bījagaṇita*, and *Siddhānta-śiromaṇi,* which acquired the status of canonical textbooks of Indian mathematics and astronomy. Upapattis (proofs) and explanations in Bhāskara's *Vāsanābhāṣyas*

Brahmagupta's Formulae for Cyclic Quadrilaterals



The diagonals e, f are given in terms of the sides a,b,c,d, by the formulae

$$e = \sqrt{\frac{(ab+bc)(ac+bd)}{ab+cd}}, \quad f = \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}$$

The area is given by

$$A = [(s-a)(s-b)(s-c)(s-d)]^{\frac{1}{2}} \ with \ \ s = \frac{(a+b+c+d)}{2}$$

VARGA PRAKṚTI

In Chapter 18 of his Brāhmasphuṭasiddhānta (c.628), Brahmagupta discusses the problem of solving for integral values of X, Y, the equation

$$\mathbf{X^2 - D\,Y^2 = K}$$

given a non-square integer $D > 0$, and an integer K.

X is called the jyeṣṭha-mūla, Y is called the kaniṣṭha-mūla
D is the prakṛti, K is the kṣepa

One motivation for this problem is that of finding rational approximations to square-root of D. If X, Y are integers such that $X^2 - D\,Y^2 = 1$, then,

$$\left| \sqrt{D} - (X/Y) \right| \leq 1/2XY < 1/2Y^2$$

The Śulva-sūtra approximation $\sqrt{2} \sim 1 + 1/3 + 1/3.4 - 1/3.4.34 = 577/408$

is an example as $(577)^2 - 2\,(408)^2 = 1$.

BRAHMAGUPTA'S BHĀVANĀ

मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाच्च।

आद्यवधो गुणकगुणःसहान्त्यघातेन कृतमन्त्यम्॥

वज्रवधैक्यं प्रथमं प्रक्षेपः क्षेपवधतुल्यः।

प्रक्षेपशोधकहृते मूले प्रक्षेपके रूपे॥

If $X_1^2 - D\,Y_1^2 = K_1$ and $X_2^2 - D\,Y_2^2 = K_2$ then

$$(X_1 X_2 \pm D\,Y_1 Y_2)^2 - D\,(X_1 Y_2 \pm X_2 Y_1)^2 = K_1 K_2$$

In particular given $X^2 - D\,Y^2 = K$, we get the rational solution

$$[(X^2 + D\,Y^2)/K]^2 - D\,[(2XY)/K]^2 = 1$$

Also, if one solution of the Equation $X^2 - D\,Y^2 = 1$ is found, an infinite number of solutions can be found, via $(X, Y) \rightarrow (X^2 + D\,Y^2, 2XY)$

CAKRAVĀLA: THE CYCLIC METHOD

The first known description of the ***cakravāla* or the cyclic method** occurs in the commentary *Sundarī* on *Laghubhāskarīya* by Udayadivākara (c.1073), who cites the relevant verses of Ācārya Jayadeva.

In his *Bījagaṇita*, Bhāskarācārya (c.1150) has given the following description of the *cakravāla* method:

ह्रस्वज्येष्ठपदक्षेपान् भाज्यप्रक्षेपभाजकान्।कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते॥
गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा।तत्तु क्षेपहृतं क्षेपो व्यस्तःप्रकृतितश्च्युते॥
गुणलब्धिःपदं ह्रस्वं ततो ज्येष्ठमतोऽसकृत्।त्यक्त्वा पूर्वपदक्षेपांश्चक्रवालमिदं जगुः॥
चतुर्द्वेकयुतावेवमभिन्ने भवतःपदे।चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना॥

By stating that this process is referred to as *cakrāvalā* (चक्रवालमिदंजगुः), Bhāskara is clearly implying that the method was well known and not invented by him.

BHĀSKARA'S EXAMPLE: $x^2 - 61\,y^2 = 1$

I	$\boldsymbol{P_i}$	$\boldsymbol{k_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\varepsilon_i}$	$\boldsymbol{x_i}$	$\boldsymbol{y_i}$
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	3	-1	164	21
4	6	5	3	1	453	58
5	9	4	4	-1	1,523	195
6	7	-3	5	1	5,639	722
7	8	-1	16	-1	29,718	3,805
8	8	-3	5	-1	469,849	60,158
9	7	4	4	1	2,319,527	296,985
10	9	5	3	-1	9,747,957	1,248,098
11	6	-5	3	1	26,924,344	3,447,309
12	9	-4	4	-1	90,520,989	11,590,025
13	7	3	5	1	335,159,612	42,912,791
14	8	1	16	-1	**1,766,319,049**	**226,153,980**

EULER-LAGRANGE AND CAKRAVALA METHOD FOR $x^2 - 61\,y^2 = 1$

	Euler-Lagrange				***Cakravāla***					
I	P_i	k_i	a_i	ε_i	P_i	k_i	a_i	ε_i	X_i	Y_i
0	0	1	7	1	0	1	8	1	1	0
1	**7**	**-12**	**1**	**1**					**7**	**1**
2	5	3	4	1	8	3	5	-1	8	1
3	7	-4	3	1	7	-4	4	1	39	5
4	**5**	**9**	**1**	**1**					**125**	**16**
5	4	-5	2	1	9	-5	3	-1	164	21
6	6	5	2	1	6	5	3	1	453	58
7	**4**	**-9**	**1**	**1**					**1070**	**137**
8	5	4	3	1	9	4	4	-1	1523	195
9	7	-3	4	1	7	-3	5	1	5639	722
10	**5**	**12**	**1**	**1**					**24079**	**3083**
11	7	-1	14	1	8	-1	16	-1	29718	3805

The steps which are skipped in *cakravāla* are highlighted

The corresponding simple continued fraction expansion is

$$\sqrt{61} = 7 + \overline{\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{14+}}$$

The nearest square continued fraction is

$$\sqrt{61} = 8 + \overline{\frac{-1}{5+}\,\frac{1}{4+}\,\frac{-1}{3+}\,\frac{1}{3+}\,\frac{-1}{4+}\,\frac{1}{5+}\,\frac{-1}{16+}}$$

DEVELOPMENT OF INDIAN ASTRONOMY & MATHEMATICS

Medieval Period (1200 – 1750)

- *Vākyakaraṇa* (c. 1300), the basic text of *Vākya* system: Computing true longitudes directly for suitable cycles of days.
- *Gaṇitasārakaumudī* (in Prākṛta) of Ṭhakkura Pherū (c.1300) and other works in regional languages such as *Vyavahāragaṇita* (Kannaḍa) of Rājāditya, *Pāvulūrigaṇitamu* of Pāvulūri Mallaṇa in Telugu, and *Cūḍāmaṇi Uḷḷamuḍaiyān* of Tirukkoṭṭiyūr Nambi in Tamil.
- *Gaṇitakaumudī* and *Bījagaṇitāvataṃsa* of Nārāyaṇa Paṇḍita (c. 1350), the most comprehensive texts on Pāṭīgaṇita and Bījagaṇita.
- *Yantrarāja* of Mahendrasūri (c.1370)
- *Makarandasāriṇī* (c.1478)
- Works of Jñānarāja (c.1500), Gaṇeśa Daivajña (b.1507), Sūryadāsa (c.1541), Kṛṣṇa Daivajña (c.1600) and Nṛsiṁha Daivajña (c.1620).
- Works of Nityānanda (c.1639), Munīśvara (c.1646) and Kamalākara (c.1658). Interaction with Islamic Astronomical Tradition.
- Mathematics and Astronomy in the Court of Savai Jayasiṁha (1686-1743). Five Observatories at Varanasi, Ujjain, Jaipur, Mathura and Delhi. Translation from Persian of Euclid and Ptolemy.

PANDIAGONAL 4x4 SQUARES OF NĀRĀYAṆA

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

Pan-diagonal Magic Square: Apart from the sum of the entries of each row, column and the principal diagonals, the sum of all the "broken diagonals" add up to the same number.

Nārāyaṇa Paṇḍita displayed 24 pan-diagonal 4x4 magic squares, with entries 1, 2, ..., 16, the top left entry being 1. Nārāyaṇa also remarked that (by permuting the rows and columns cyclically) we can construct 384 pan-diagonal 4x4 magic squares with entries 1, 2, ..., 16.

The fact that there are only 384 pan-diagonal 4x4 magic squares, was proved by B.Rosser and R.J.Walker in 1938. A simpler proof was given by T.Vijayaraghavan in 1941.

PROPERTIES OF PANDIAGONAL 4x4 MAGIC SQUARES

Property 1: Let M be a pan-diagonal 4x4 magic square with entries 1, 2, ..., 16, which is mapped on to the torus by identifying opposite edges of the square. Then the entries of any 2x2 sub-square formed by consecutive rows and columns on the torus add up to 34.

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

For example, 1+12+15+6 = 1+12+14+7 = 34

Property 2: Let M be a 4x4 pan-diagonal magic square with entries 1, 2, ..., 16, which is mapped on to the torus. Then, the sum of an entry of M with another which is two squares away from it along a diagonal (in the torus) is always 17.

For example, 1+16 = 6+11 = 15+2 = 4+13 = 14+3 = 9+8 = 17

PROPERTIES OF PANDIAGONAL 4x4 MAGIC SQUARES

The "neighbours" of an element of a 4x4 pan-diagonal magic square (which is mapped on to the torus as before) are the elements which are next to it along any row or column. For example, 3, 5, 2 and 9 are the "neighbours" of 16 in the magic square below.

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

Property 3 (Vijayaraghavan): Let M be a 4x4 pan-diagonal magic square with entries 1, 2, ..., 16, which is mapped on to the torus. Then the neighbours of the entry 16 have to be 2, 3, 5 and 9 in some order.

We can use the above properties to construct 4x4 pan-diagonal magic squares starting with 1 placed in any desired cell.

Proposition: There are precisely 384 pan-diagonal 4x4 magic squares with entries1, 2, ..., 16.

Nārāyaṇa Paṇḍita on *Vārasaṅkalita* (c.1350)

Āryabhaṭīya, gives the sum of the sequence of natural numbers

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

as also the result of first order repeated summation:

$$\frac{1.2}{2} + \frac{2.3}{2} + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Āryabhaṭa's result for repeated summation was generalised to arbitrary order by Nārāyaṇa Paṇḍita (c.1350). Let

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = V_n^{(1)}$$

Then, **Nārāyaṇa**'s result is

$$\begin{aligned} V_n^{(r)} &= V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \\ &= \frac{[n(n+1)\ldots(n+r)]}{[1.2\ldots(r+1)]} \end{aligned}$$

DEVELOPMENT OF INDIAN ASTRONOMY & MATHEMATICS

Medieval Period (1200 – 1750)

- Mādhava (c.1400): Founder of the Kerala School. Infinite series for π, sine and cosine and fast convergent approximations to them. *Veṇvāroha*, *Lagnaprakaraṇa, Agaṇitagrahacāra*
- Parameśvara (c.1380-1460): Over 55 years of observations.
- Nīlakaṇṭha Somayājī (c.1444-1540): Revised planetary model.
- Exposition of Mathematics and Astronomy with proofs in *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530), *Kriyākrmakarī* and *Yuktidīpikā* of Śaṅkara Vāriyar (c.1540).
- Works of Acyuta Piṣāraṭi (c.1550-1621), Putumana Somayājī (c.1600)

Modern Period (Post 1750)

- Śaṅkaravarman (1784-1839): *Sadratnamāla*
- Candraśekhara Sāmanta (1835-1904): *Siddhāntadarpaṇa* (all three lunar inequalities)
- Srinivasa Ramanujan (1887-1920), Harish Chandra (1923-1983), ...
  Srinivasa Varadhan (b.1940), Manjul Bhargava (b.1974), Akshay Venkatesh (b. 1981), ...
- Meghnad Saha (1893-1956), Subrahmanyan Chandrasekhar (1910-1995), ...

A HISTORY OF EXACT RESULTS FOR π

Mādhava (1375)	$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$ $\pi/\sqrt{12} = 1 - 1/3.3 + 1/3^2.5 - 1/3^3.7 + \ldots$ $\pi/4 = 3/4 + 1/(3^3-3) - 1/(5^3-5) + 1/(7^3-7) - \ldots$ $\pi/16 = 1/(1^5+4.1) - 1/(3^5+4.3) + 1/(5^5+4.5) - \ldots$
Francois Viete (1593)	$2/\pi = \sqrt{[1/2]}\ \sqrt{[1/2 + 1/2\sqrt{(1/2)}]}\ \sqrt{[1/2 + 1/2\sqrt{(1/2+1/2\sqrt{(1/2)})}]}\ldots$ (Infinite product)
John Wallis (1655)	$4/\pi = (3/2)(3/4)(5/4)(5/6)(7/6)(7/8)\ldots$ (Infinite product)
William Brouncker (1658)	$4/\pi = 1 + 1^2/2 + 3^2/2 + 5^2/2 + \ldots$ (Continued fraction)
Isaac Newton (1665)	$\pi = 3\sqrt{3}/4 + 24\,[1/3.8 - 1/5.32 - 1/7.128 - 1/9.512 - \ldots]$
James Gregory (1671)	$\tan^{-1}(x) = 1 - x/3 + x^2/5 - \ldots$
Gottfried Leibniz (1674)	$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$
Abraham Sharp (1699)	$\pi/\sqrt{12} = 1 - 1/3.3 + 1/3^2.5 - 1/3^3.7 + \ldots$
John Machin (1706)	$\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$

Ramanujan (1910, 1914)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

SOPHISTICATED METHODOLOGY OF INDIAN MATHEMATICS

- Kerala mathematics in the fourteenth to sixteenth century. This culture was the most advanced mathematical culture of its time throughout the world. Its breakthroughs include power series like approximations of trigonometric functions and the value of pi. …
- Some historians of mathematics claimed that Indian mathematics had no interest in proofs. This position has been refuted, as discussions of proofs and justification have been documented already in early classical Sanskrit mathematics. …Within this tradition, the most highly elaborate proofs that survive in Indian sources come from Kerala. …
- Indian proofs did not start with axiomatic foundations. Indian proofs could rely on evidence elicited from observing a diagram or the use of analogies to physical situations.

Roy Wagner, "Does mathematics need Foundations" in S. Centrone *et al* ed., *Reflections on Foundations of Mathematics*, Springer 2019, pp.390-391

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

The *National Curriculum Framework for School Education 2003* (NCFSE) prescribes that: “The rich history of Indian contributions to various fields (also referred to as Indian Knowledge Systems) [be] incorporated throughout the curriculum, as this not only develops pride and self-confidence, but also enriches learning in those areas” (NCFSE, p.144).

We present a set of Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum, as compiled by scholars of IKS

1. While discussing any topic, algorithm or result, it would be pertinent to mention the pioneering work done by Indian mathematicians in that area. For instance, while discussing the standard method of extracting square root (say in Grade 8), it may be explained that it is a slight modification and improvement of the method first presented in the text *Āryabhaṭīya* of Āryabhaṭa.

2. Apart from introducing specific Indian contributions, it is also important to present some of the illustrative examples given in the classical texts of Indian mathematics (like problems on linear and quadratic equations in the text *Līlāvatī* of Bhāskarācārya and *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita).

3. While introducing any of the Indian contributions, it is important to ensure that they are not presented as a separate or supplementary part of the text, but as an integral part of the corresponding section/chapter (Arithmetic, Algebra, Geometry, Trigonometry, etc.) where similar or related topics are discussed. Presenting IKS related material in the form of boxes (which has been the practice so far) may be avoided as far as possible.

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

Topics that may be introduced in the Foundational Stage (Till Grade 2)

- Numerals and number names from one to 100 in the Regional Language and Hindi/Sanskrit.
- Addition tables (for single digits).

Topics that may be introduced in the Preparatory Stage (Grades 3-5)

- Number names to one crore in Regional Language, Hindi/Sanskrit.
- Multiplication tables. [Knowing the addition and multiplication tables and reciting them in the class have been an integral part of our indigenous pedagogy, which seems to have ensured universal basic numeracy.]
- Indian origin of zero and the place value system of representing numbers. The use of place value system in arithmetical calculations.
- Some elementary examples of magic squares (3x3 and 4x4 squares), Kolam patterns and traditional board games..

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

Special Topics to be Introduced in the Middle Stage (Grades 6-8)

Arithmetic

- A historical overview of the development of place value system in India starting from the use of oral decimal system for naming numbers in Sanskrit from the Vedic times. Correspondence between powers of ten and place value. The efficiency and power of the use of place value system in arithmetical calculations.

- Indian origin of the rules of operations with fractions (with numerator larger than 1)

- Standard method of calculating the square-root and cube root (which is an improvement of the method outlined first by Āryabhaṭa)

- Rule of three (Trairāśika) and its applications to calculation of interest etc. Inverse rule of three. Generalisation to rule of five etc. Importance given to the rule of three in India and later (as the "Golden Rule") in Europe.

- Traditional Indian method for constructing magic squares of odd-order.

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

Algebra

- Brahmagupta's rules for arithmetic operations with negative numbers and zero.

- Indian method of representing unknowns by symbols (using letters of the alphabet). Brahmagupta's rules for standard operations with algebraic polynomials.

- Analogy between decimal system and polynomials (Newton's quotation). The algebraic nature of the place value notation and how it can be used to justify all the standard algorithms used in arithmetic.

Geometry

- Baudhāyana-Pythagoras theorem and its applications given in several Indian texts. Some Indian proofs of Baudhayana-Pythagoras Theorem

- Computing the base intercepts of a triangle and the perpendicular to the base, given the three sides of a triangle.

- Some of the exact geometrical constructions presented in the *Śulbasūtras* and their algebraic basis.

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

Special Topics to be Introduced in the Secondary Stage (Grades 9-10)

Arithmetic

- Comparison with other systems such as the Mesopotamian, Greek and Roman numerals. Importance of one symbol for one digit, which is the characteristic of the Indian system. Transmission of the Indian place value system to the rest of the world (Quotes from Severus Sebokht, Ibn Sina, Leonardo Pisano (Fibonacci) and Simon Laplace).
- Rational approximation of irrational numbers. Śulbasūtra approximation for the square-root of 2. Bakhśāli formula for computing approximate square-roots. Śrīdhara's method of obtaining rational approximations of square-roots based on a modification of the Āryabhaṭa method.

Algebra

- Piṅgala's algorithm for the *n*-th power of a number.
- Āryabhaṭa formulae for the sum and repeated sum of natural numbers, and for the sum of squares and cubes of natural numbers. Various identities in Śulva-sūtras and works of Brahmagupta and Śrīdhara.
- Importance of symbols and equations can be explained by comparing them with the Regula Falsi (Rule of False Position).
- Brahmagupta's method of elimination for solving linear equations in several unknowns.
- Brahmagupta's formula for solving a quadratic equation. Śrīdhara's method of completing the square.

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

Geometry

- Āryabhaṭa's approximate value of π and its accuracy to four decimal places and his emphasis that this is "āsanna" (approximate). Later scholars such as Āryabhaṭa II and Bhāskarācārya refer to the value 22/7 as "sthūla" (coarse)

- Brahmagupta's formula for the circum-radius of a triangle. Brahmagupta's formulae for the diagonals and the area of a cyclic quadrilateral.

Trigonometry

- Indian use of degrees, minutes etc. to measure arcs of a circle. Āryabhaṭa's definition of the Jyā of an arc, its transmission to West Asia and how it is related to the sine function that is used to-day. How the Jyā or sine function enabled Indian astronomers to express all the Astronomical relations by simple formulae.

- Āryabhaṭa's sine table.

- The approximate formula for sine given by Bhāskara I and its accuracy. Applications of the formula for computing chords and sides of regular polygons inscribed in a circle.

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

Senior Secondary Stage (Grades 11-12)

Some of the following topics from Indian mathematical texts can be introduced as part of any course on Mathematics where they fit in. There can also be a course titled "Development of Mathematics in India: Some Highlights", where most of the topics noted below can be included.

- An overview of development of mathematics in India
- The Bhūtasaṅkhyā and Kaṭapayādi systems (and the Tamil Vowel-based system) of numeration to denote numbers in Indian literature, inscriptions etc., both in Sanskrit and other Indian Languages.
- Mahāvīra's formulae for the sum of squares and cubes of an arithmetic sequence. Nārāyaṇa's formula for higher order repeated summations (Vārasaṅkalita) of an arithmetic sequence and its applications.
- Combinatoric tools (Pratyayas) of Piṅgala. Prastāra or enumeration of Varṇa-vṛttas (syllabic metres). Naṣṭa and Uddiṣṭa (ranking and un-ranking) using the binary representation of numbers. Meruprastāra and computation of binomial coefficients. Mahāvīra's formula for binomial coefficients.
- Prastāra of Mātrā-vṛttas (moric metres) and Virahāṅka-Fibonacci numbers and their generalisation by Nārāyaṇa Paṇḍita.Prastāra of Tānas (permutations of musical notes) and Tālas (rhythms) discussed by Śārṅgadeva. General treatment of combinatoric problems by Nārāyana. Prastāra of permutations with repetitions. Prastāra of combinations.

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

- Mathematical study of magic squares by Nārāyaṇa. Classification of magic squares. Use of arithmetic sequences.
- Construction of 4x4 pan-diagonal magic squares in Indian tradition (Nāgārjuna, Varāhamihira, Bhaṭṭotpala, Nārāyaṇa Paṇḍita, Vijayaraghavan). Folding method of constructing magic squares of doubly-even and odd orders.
- Linear indeterminate equations and their solution by the Kuṭṭaka method (as discussed say in *Āryabhaṭīya and Līlāvatī*). Applications. Samsliṣṭa-kuṭṭaka or the Indian method of solving the Chinese remainder problem.
- Second order indeterminate equation (Vargaprakṛti) introduced by Brahmagupta. Bhāvana principle of Brahmagupta. Rational solutions Vargaprakṛti equation with Kṣepa 1 (the Pell's equation) due to Śrīdhara and Śrīpati. Applications to rational approximations of square-roots.
- Cakravāla method of solution of Vargaprakṛti equation with Kṣepa 1 (the Pell's equation) due to Jayadeva. Its modification due to Nārāyaṇa.
- Krishnaswami Ayyangar's formulation of Cakravāla in terms of semi-regular continued fractions. Optimality of the Cakravāla method when compared with the currently well-known the Euler-Lagrange method.

Suggestions for Incorporating Indian Mathematical Contributions in School Curriculum

- The third diagonal of a cyclic quadrilateral. Parameśvara formula for the circumradius of a cyclic quadrilateral. Brahmagupta and Bhāskara II method of constructing cyclic quadrilaterals from a pair of right-angled triangles.
- Āryabhaṭa's formula for the second order sine differences in the form discussed by Nīlakaṇṭha – the discrete version of harmonic equation.
- The notion of instantaneous velocity and use of derivatives in Indian astronomy (Muñjāla, Bhāskara II, Nīlakaṇṭha and Acyuta).
- Mādhava's infinite series for π. Its proof using binomial expansion and mathematical induction (for estimating the asymptotic behaviour of the sum of powers of natural numbers) as given in *Yuktibhāṣā* of Jyeṣṭhadeva. Mādhava's end-correction terms and transformed series with faster convergence. Mādhava's approximation to π accurate to 11 decimal places.
- Mādhava's arctan, sine and cosine series. Proofs of these series as given in *Yuktibhāṣā*. Mādhava's efficient method of computation of sines for arbitrary arcs/angles.
- *Yuktibhāṣā* proofs of properties of cyclic quadrilateral and sphere.
- Some remarks on the "constructive approach" followed in the Indian mathematical tradition.
- A brief introduction to the notable contributions by Indian mathematicians in the twentieth and twenty-first centuries.

References

1. B. Datta and A. N. Singh, *History of Hindu Mathematics*, 2 Parts, Lahore 1935, 1938; Reprint, Asia Publishing House, Bombay 1962. Reprint, Bharatiya Kala Prakashan, Delhi 2004.
2. C. N. Srinivasa Iyengar, *History of Indian Mathematics*, The World Press, Calcutta 1967.
3. A. K. Bag, *Mathematics in Ancient and Medieval India*, Choukhambha, Varanasi 1979.
4. T. A. Saraswati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarsidass, Varanasi 1979.
5. Kim Plofker, *Mathematics in India*, Princeton University Press, Princeton, 2009. Reprint, Hindustan Book Agency, New Delhi 2012.
6. G. G. Joseph, *Indian Mathematics Engaging with the World from Ancient to Modern Times*, World Scientific, New Jersey 2016.
7. P. P. Divakaran, *The Mathematics of India Concepts, Methods, Connections*, Hindustan Book Agency, New Delhi 2018. Reprint, Springer, New York 2018
8. K. Ramasubramanian, M. S. Sriram and M. D. Srinivas, NPTEL Course on 'Development of Mathematics in India from Vedic Period to Modern Times', I.I.T. Madras 2013 (A course of forty lectures available at http://nptel.ac.in/courses/111101080/).