Jyotpatti : Trigonometry in India

From Āryabhaṭa to Bhāskara II to Nityānanda

Professor K. Ramasubramanian
Department of Humanities IIT Bombay

and

Professor Clemency Montelle
School of Mathematics and Statistics University of Canterbury

HOMI Workshop IIT Gandhinagar

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

Outline of the talk

Here is an overview of today's presentation:

- Rationale and motivation for the study of trigonometry
- Underlying geometry and technical terms
- Various techniques for computing Sines from key authors

Rationale and motivation

Introduction

What should our attitude be while looking at the past?

- Right at the beginning of his famous drama *Mālavikāgnimitram* (1.2) (through the words of *sūtradhāra*) Kālidāsa observes:

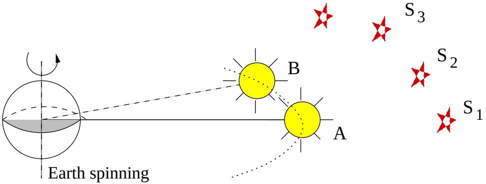
  पुराणमित्येव न साधु सर्वं न चापि काव्यं नवमित्यवद्यम्।
  सन्तः परीक्ष्यान्यतरद्भजन्ते मूढः परप्रत्ययनेयबुद्धिः॥

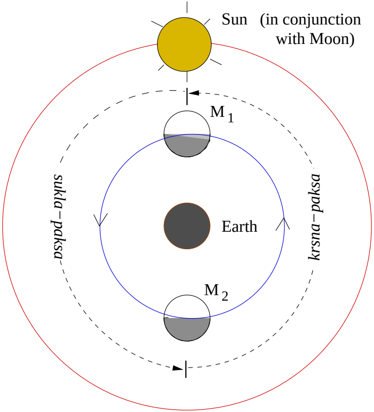
  By virtue of being ancient, not all is good. And [similarly], by virtue of being new, modern literature should not be condemned. The noble ones having examined, choose either of them; the unintelligent is carried away by the views of others.

- Mindless glorification of the past $\rightsquigarrow$ counterproductive.
- Equally counterproductive would be downright condemnation.
- It has become fashionable to build and spin theories as if looking into the past is $\equiv$ 'taking back to stone age', 'a great step backwards' etc.
- Indians have never shied to break the moulds of the past!

Motivation for finding accurate sine values

- The positions of the planets in the background of stars forms the basis for reckoning time.
- Determination of their positions crucially depends upon accurate knowledge of sine function.





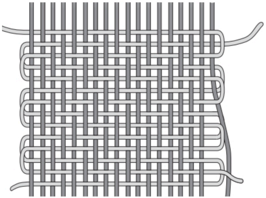
Bhāskara's (b. 1114 CE) metaphor for sines and cosines

Opening verse from Bhāskara's treatment of trigonometry in the *Siddhāntaśiromaṇi*

> पटो यथा तन्तुभिरूर्ध्वतिर्यक्-
> रूपैर्निबद्धोऽत्र तथैव गोलः ।
> दोःकोटिजीवाभिरमुं प्रवक्तुं
> ज्योत्पत्तिमेव प्रथमं प्रवक्ष्ये ॥१॥
>
> *paṭo yathā tantubhir ūrdhvatiryak-*
> *rūpair nibaddho'tra tathaiva golaḥ |*
> *doḥkoṭijīvābhir amuṃ pravaktuṃ*
> *jyotpattim eva prathamaṃ pravakṣye ||1||*
>
> *Just as fabric (paṭa) is made of warp and weft threads, so too [the study of] spherics is criss-crossed with sines and cosines. [Therefore,] I will firstly narrate the jyotpatti ('generation of sines') to elaborate on this.*

A textile analogy to mathematical sines and cosines

Just as fabric (*paṭa*) is made of warp and weft threads, so too [the study of] spherics is criss-crossed with Rsines and Rcosines. [Therefore,] I will firstly narrate the *jyotpatti* (‘generation of sines’) to elaborate on this.



What is the significance of this analogy?

Nityānanda (17c) on the importance of trigonometry

आचार्यवर्या गणकास्त एव
जानन्ति ये ज्यानयनोपपत्तिम् ।
ततोऽधिगन्तुं पदवीं च तेषां
महाजडो वाञ्छति मादृशोऽपि ॥ १९ ॥

ācāryavaryā gaṇakās ta eva
jānanti ye jyānayanopapattim |
tato 'dhigantuṃ padavīṃ ca teṣāṃ
mahājaḍo vāñchati mādṛśo 'pi ||19||

Oh revered teachers! Only those are considered mathematicians who know the rationale behind the computation of Sines. Therefore, [by venturing to explain that,] even a dullard like me wishes to accomplish their status.

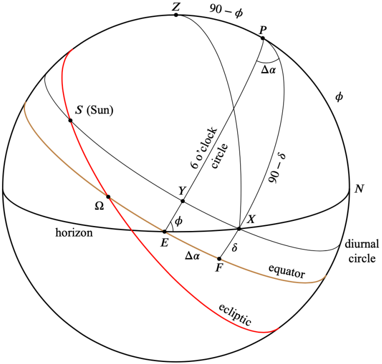
(*Sarvasiddhāntarāja* verse 1)

Nityānanda on the importance of knowing the rationale

यत्किंचिद्गणितं ग्रहस्य गणको जानान्विना वासनां
पृष्टः सन्नपरेण गोलविदुषा प्रश्नप्रपञ्चोक्तिभिः ।।
किं ब्रूते प्रतितं तदुत्तरमयं गोलागमाज्ञो यत-
स्तस्माद्गोलविचारचारुरचनां वक्ष्ये सतां प्रीतये ।।

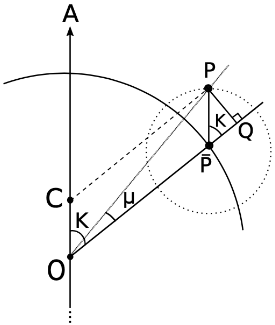
A mathematician knowing some mathematics concerning planets but without [knowing] the rationale was being questioned by another person who was an expert in spherics. What could he say as an answer to those [questions], being ignorant in the science of the spheres? Because of this, I will state a pleasing composition on the investigation of the spheres for the delight of all beings. (Gola, verse 2)

Sines and cosines are everywhere in astronomy!



(a) Ascensional differences

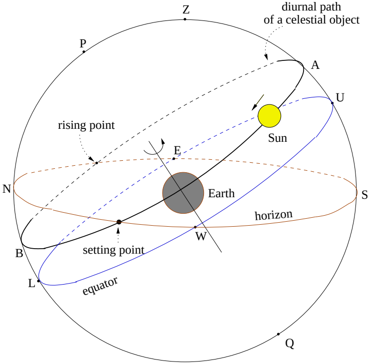
$$\Delta\alpha = \sin^{-1} \frac{\sin\phi \cdot \sin\delta}{\cos\phi \cdot \cos\delta}$$



(b) Epicyclic corrections to planetary longitudes

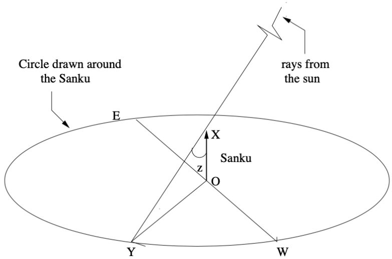
$$\sin\mu = \frac{r_M \sin\kappa_M}{R}$$

Sines and cosines are everywhere in astronomy!



(a) Solar declination

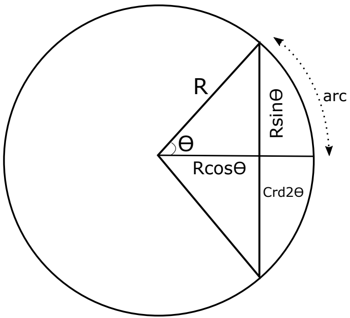
$\sin\delta = \sin\epsilon \cdot \sin\lambda$

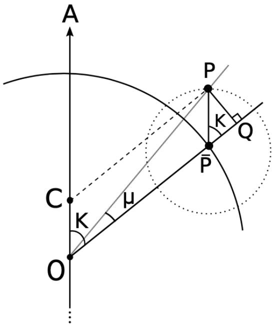


(b) Zenith distance and the length of the shadow

$t = R\sin^{-1}\left[\frac{R\cos z}{\cos\phi\cos\delta} \pm R\sin\Delta\alpha\right] \mp \Delta\alpha.$

Arcs and chords: the origins of trigonometry in India



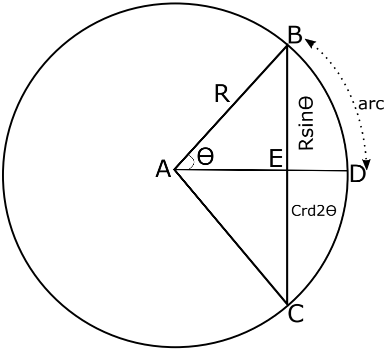


यतो ज्यकार्धाग्रगतो ग्रहेन्द्रः तिर्यक्स्थितो मध्यमसूत्रतः स्यात् ॥

This is because [in astronomical models, the position of] the planet, lying at the tip of the half-chord, situated perpendicular to the central line [intersecting the chord, is computed using this measure].

Nityānanda *Sarvasiddhāntarāja* (17 cent.)

Technical terminology is introduced: bows and bow strings



Sanskrit	Translation	Math	Figure
चाप	bow	arc	BDC
ज्या	bow-string	chord	BEC
ज्यार्ध	half the bow-string	sine	BE or EC
शर	arrow	versed sine	ED

The basis for the terminology

The 'bow-string' and its etymological journey

ज्या   जीवा

(Sanskrit: *jyā* or *jīvā* "bow-string")

$\Downarrow$

الجيب

(Arabic: *al-jaib* "bay, pocket, fold, cavity")

$\Downarrow$

sinus

(Latin: "bay, cavity")

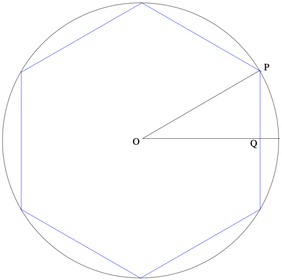
The approach in early *siddhāntas*:

Varāhamihira's *Pañcasiddhāntikā*

Sixth Century

Building a sine table: Varāhamihira's *Pañcasiddhāntikā*

Varāhamihira uses basic geometric relations of polygons inscribed in a sphere to determine key sine values.



$$
\begin{aligned}
R\sin(30^\circ) &= \frac{R}{2} \\
R\sin(45^\circ) &= \frac{R}{\sqrt{2}} \\
R\sin(60^\circ) &= \frac{\sqrt{3}}{2}R \\
R\sin(90^\circ) &= R
\end{aligned}
$$

Building a sine table: Varāhamihira's *Pañcasiddhāntikā*

Using $R = 120$, and several extra identities, he produces 24 sine values.

arc	sine
3;45	$7'51$
7;30	$15'40$
11;15	$23'25$
15	$31'4$
18;45	$38'34$
22;30	$45'56$
26;15	$53'5$
30	$60'0$
$\vdots$	$\vdots$
75	$115'55$
78;45	$117'42$
82;30	$118'59$
86;15	$119'44$
90	$120'0$

$$R\sin(\theta) = R\cos(90-\theta)$$

$$R\sin^2(\theta) + R\cos^2(\theta) = R^2$$

$$R\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{R\sin^2(\theta) + R\operatorname{vers}^2(\theta)}}{2}$$

Building a sine table: Varāhamihira's *Pañcasiddhāntikā*

- The following verses in the *Pañcasiddhāntikā* present the values:

  शेषज्याः स्वरतिथयो गुण-शिव-घृति'भिश्च 'विंशतिः' सहिता ।
  'पञ्चनरकं' शतार्धं त्रिसमेतं 'षष्टि'रिति लिप्ता ॥
  सैकाजे पञ्चाशत् 'पञ्चाष्टकं' 'पञ्चवर्गवेदा'श्च ।
  'त्रिंशच्चतुर्भिरधिका' 'षट्-पञ्चाशच्छराः'[1] शून्यम् ॥

- We list the values given above in the form of a table.

No.	1	2	3	4	5	6	7	8
Arc	$3^\circ 45'$	$7^\circ 30'$	$11^\circ 15'$	$15^\circ 0'$	$18^\circ 15'$	$22^\circ 30'$	$26^\circ 15'$	$30^\circ 0'$
Sine	$7'51''$	$15'40''$	$23'25''$	$31'4''$	$38'34''$	$45'56''$	$53'5''$	$60'0''$

Note: The phrase सैकाजे पञ्चाशत् should be understood as सैका पञ्चाशत् अजे (=मेषे). It is also interesting to note the use of शून्यम् here as a number , and not merely as place holder.

- This is one of the earliest instances where we fine a very interesting way of presenting the tabular values.

[1]There is a typo in the printed version. It reads as – षट्-पञ्चशच्छराः।

Defining Sines recursively: The *Paitāmahasiddhānta*

Defining sines recursively: the *Paitāmahasiddhānta* (early 5th century (?))

"The first Sine is a ninety-sixth part of 21,600. If one divides the first Sine by the first Sine and subtracts the quotient from the first Sine, one obtains the difference of the second Sine; the sum of the first Sine and the difference of the second Sine is the second Sine. If one divides the second Sine by the first Sine and subtracts the quotient from the difference of the second Sine, one obtains the difference of the third Sine; the sum of this and the second Sine is the third Sine . . . [And so on up to the twenty-fourth and final Sine.]

The minutes [in the argument of arc] are to be divided by 225; the Sine corresponding to the resulting serial number is to be put down. One should multiply the remainder by the difference of the next Sine and divide [the product] by 225. The sum of the quotient and the Sine which was put down is the desired Sine."

Defining sines recursively: the *Paitāmahasiddhānta*

"The first Sine is a ninety-sixth part of 21,600."

Here, the first Sine is thus:

$$\mathrm{Sin}_1 = \frac{21,600}{96} = 225'$$

Defining sines recursively: the *Paitāmahasiddhānta*

"If one divides the first Sine by the first Sine and subtracts the quotient from the first Sine, one obtains the difference of the second Sine; the sum of the first Sine and the difference of the second Sine is the second Sine. If one divides the second Sine by the first Sine and subtracts the quotient from the difference of the second Sine, one obtains the difference of the third Sine; the sum of this and the second Sine is the third Sine . . . [And so on up to the twenty-fourth and final Sine.]"

From this, a general recursive formula can be stated:

$$\mathrm{Sin}_i - \mathrm{Sin}_{(i-1)} = \mathrm{Sin}_{(i-1)} - \mathrm{Sin}_{(i-2)} - \frac{\mathrm{Sin}_{(i-1)}}{\mathrm{Sin}_{(1)}}$$

Defining sines recursively: the *Paitāmahasiddhānta*

"If one divides the first Sine by the first Sine and subtracts the quotient from the first Sine, one obtains the difference of the second Sine; the sum of the first Sine and the difference of the second Sine is the second Sine. "

Sin_2 :

$$\begin{aligned} \mathrm{Sin}_2 - \mathrm{Sin}_1 &= \mathrm{Sin}_1 - \frac{\mathrm{Sin}_{(1)}}{\mathrm{Sin}_{(1)}} \\ \mathrm{Sin}_2 - 225 &= 225 - \frac{225}{225} \\ \mathrm{Sin}_2 &= 225 - 1 + 225 \\ &= 449 \end{aligned}$$

Defining sines recursively: the *Paitāmahasiddhānta*

"If one divides the second Sine by the first Sine and subtracts the quotient from the difference of the second Sine, one obtains the difference of the third Sine; the sum of this and the second Sine is the third Sine . . . [And so on up to the twenty-fourth and final Sine.]"

Sin_3 :

$$
\begin{aligned}
\mathrm{Sin}_3 - \mathrm{Sin}_2 &= \mathrm{Sin}_2 - \mathrm{Sin}_1 - \frac{\mathrm{Sin}_{(2)}}{\mathrm{Sin}_{(1)}} \\
\mathrm{Sin}_3 - 449 &= 449 - 225 - \frac{449}{225} \\
\mathrm{Sin}_2 &= 224 - 2 + 449 \\
&= 671
\end{aligned}
$$

Defining sines recursively: the *Paitāmahasiddhānta*

If one continues, there results:

$$\mathrm{Sin}_{24} = R = \mathrm{Sin}(90^\circ) \approx 3359$$

whereas it was probably mean to yield

$$R = 3438$$

where

$$3438 \approx \frac{C}{2\pi}$$

when $C = 21,600$ arcminutes and $\pi = 3.1416$
This is analogous to the modern radian measure in which the same units are used for arc and their corresponding line segments.

Āryabhaṭa's *Āryabhaṭīya*

Two approaches for finding sine values

Encoding numerical values: Āryabhaṭa's table of sines

Āryabhaṭa composes an unusual verse encoding numerical values:

> मखि भकि फखि धखि णखि ञखि ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।
> घ्लकि किग्र हक्य धकि किच स्ग झश ङ्व क्ल प्त फ छ कलार्धज्याः ॥
>
> *makhi bhaki phakhi dhakhi ṇakhi ñakhi ṅakhi hasjha skaki kiṣga śghaki kighva* |
> *ghlaki kigra hakya dhaki kica sga jhaśa ṅva kla pta pha cha kalārdha-jyāḥ* ||
>
> *225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7 are the sine [differences] in minutes of arc.*

Āryabhaṭīya (499 CE)

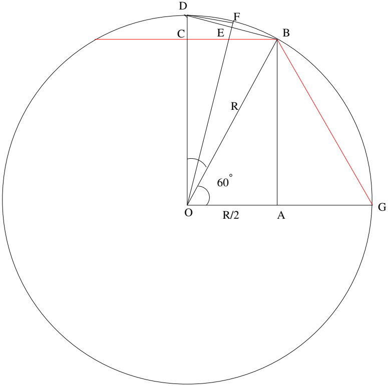
Listing sine-differences: Āryabhaṭa's *Āryabhaṭīya* (499 CE)

arc	sine	sine diff.
3;45	225	225
7;30	449	224
11;15	671	222
15	890	219
18;45	1105	215
22;30	1315	210
26;15	1520	205
30	1719	199
⋮	⋮	⋮
75	3321	65
78;45	3372	51
82;30	3409	37
86;15	3431	22
90	3438	7

- 24 values for arc-multiples of 3;45°
- Values tabulated are sine differences
- $R = 3438$
- First value: $R\sin 225' = 225$ based on the assumption that for small $\theta : R\sin\theta \approx \theta$
- Two approaches:
  - Geometric approach
  - Ingenious recursive algorithm to produce successive sine differences

Finding tabular sines: Geometrical approach

समवृत्तपरिधिपादं छिन्द्यात् त्रिभुजाच्चतुर्भुजाच्चैव ।
समचापज्यार्धानि तु विष्कम्भार्धे यथेष्टानि ।।



- We know, chord $60^\circ = R = 3438$
- From the triangle OBC,

$$R \sin 30 = BC = R/2 = 1719$$

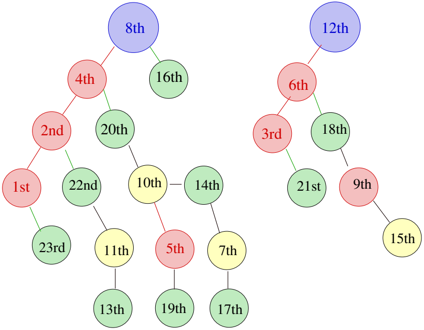
- Using *bhujā-koṭi-karṇa-nyāya* from 8th Rsine (30°) we can get 16th Rsine.
- In other words, $R \sin\theta \rightsquigarrow R\sin(90-\theta)$.
- Further, noting that

$$R\sin\theta \;\rightsquigarrow\; R\cos\theta \;\rightsquigarrow\; R \text{ vers } \theta$$
$$R\sin\theta \;\&\; R \text{ vers } \theta \rightsquigarrow R\sin\frac{\theta}{2},$$

all the 24 Rsines can be calculated.

Finding tabular sines: Geometrical approach (contd.)

- Most of the Indian astronomers have presented their sine tables by dividing the quadrant (90°) into 24 parts.
- By the principle outlined above, it can be easily shown that all the 24 Rsines can be obtained provided the 24th, 12th and 8th Rsines are known.



- The circumference of the circle was taken to be 21600 units.
- From that using the approximation for π given by him, we get $R = 24th$ Rsine ≈ 3438.
- Once this it known, by the proposed scheme of constructing the table, all that is required technically is extraction of square root, for which Āryabhaṭa had clearly evolved an efficient algorithm.

Generating sine differences: Āryabhaṭa's *Āryabhaṭīya*

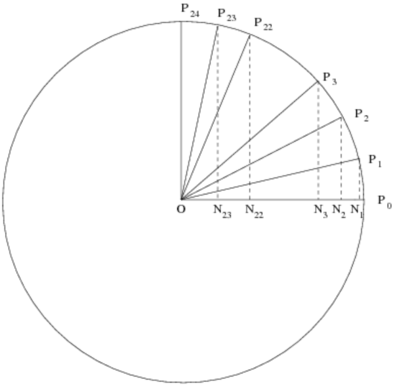


Figure: 24 sines in the quadrant

Ingenious recursive algorithm to produce successive sine values

$$\Delta_{i+1} - \Delta_i = -\frac{J_i}{J_1}$$

where

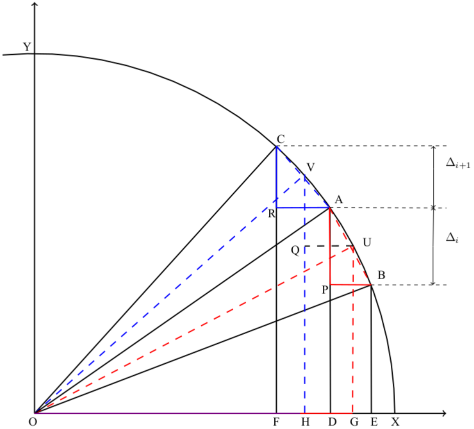
$$J_i = R\sin(i \times 225) = P_i N_i,$$

and

$$\Delta_i = J_i - J_{i-1}.$$

Derivation of Āryabhaṭa's recursive relation

- The recursive relation given by Āryabhaṭa can be derived by considering the following figure, and employing simple mathematical principles.



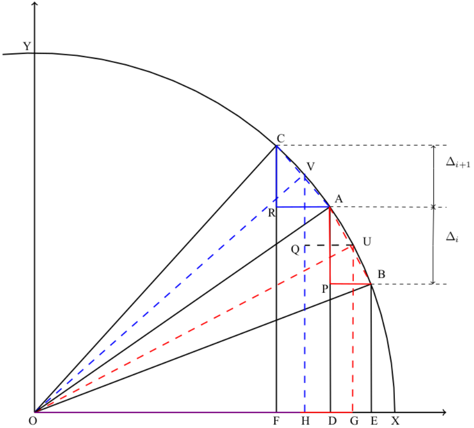
- In the quadrant OXY, $AB = AC = \alpha = 225'$
- Let $AX = i\alpha$, $BX = (i-1)\alpha$, $CX = (i+1)\alpha$
- The points U & V are such that $AU = BU = \frac{1}{2}\alpha$ & $AV = VC = \frac{1}{2}\alpha$
- The points P, Q & R are the feet of the perpendiculars drawn from B, U, & A on to AD, VH, & CF.
- What we need to find is: $AP - CR = \Delta_i - \Delta_{i+1}$

- The expression for this second order sine-difference, is obtained primarily by comparing similar triangles and writing down a few equations.

Derivation of Āryabhaṭa's recursive relation

- Here $\triangle APB$ & $\triangle OGU$ are similar. Hence, $AP = \dfrac{AB}{OU} \times GO$. Similarly, from $\triangle CRA$ & $\triangle OHV$, we have $CR = \dfrac{CA}{OV} \times OH$. Finding the difference of these two we get,

$$AP - CR = \frac{AB}{OU}(OG - OH) = \frac{AB}{OU} \times HG \qquad (1)$$



- From $\triangle VQU$ & $\triangle ODA$, we have

$$QU = \frac{VU}{OA} \times AD \qquad (2)$$

- Now, $HG = QU$. Using this in (1) we get

$$AP - CR = \left(\frac{AB}{OU}\right)^2 \times AD \qquad (3)$$

- From $\triangle APB$, $AB = 2R\sin\frac{\alpha}{2}$. Also, $AD = J_i = R\sin i\alpha$. Thus we have,

$$\Delta_i - \Delta_{i+1} = \left(2\sin\frac{\alpha}{2}\right)^2 \times J_i \qquad (4)$$

Āryabhaṭa's table for computing Rsines

- Using either/both the approaches, Āryabhaṭa could have obtained the Sine values.Then he uses his ingenuity to present them in the form of a verse in *Gītikā-pāda* of *Āryabhaṭīya* (verse 12).
- This verse[2] lists the 24 first order Rsine-differences (in arc-minutes):
  मखि भखि फखि धखि णखि ञखि
  ङखि हस्झ स्ककि किष्ग स्घकि किघ्व । (should be किषग सघकि ??)
  घ्लकि किग्र हक्य धकि किच
  स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्याः ।।

  225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225′ of arc] in terms of the minutes of arc.
- In Āryabhaṭa's notation: म $\rightarrow$ 25; & ख $\rightarrow$ 200;

[2] This verse is one of the most terse verses in the entire Sanskrit literature that one ever comes across. Only after several trials would it be ever possible to read the verse properly, let alone deciphering its content.

Abbreviating Sine tables:

Brahmagupta's *Khaṇḍakhādyaka*

665 CE

Abbreviating sine tables: Brahmagupta's *Khaṇḍakhādyaka* (665)

- Brahmagupta's *karaṇa* text provides a much shorter verse encoding only six numbers using the 'object-numeral' system to present Sines:

  त्रिंशत्सनवरसेन्दुर्जिनतिथिविषयाश्च राश्यर्धचापानाम् ।
  अर्धज्याखण्डानि ज्याभुजैक्यं सभोज्यफलम् ।।

  triṃśatsanavarasendurjinatithiviṣayā ca rāśyārdhacāpānām
  ardhajyākhaṇḍāni jyābhuktaikyaṃ sabhogyaphalam ||

  Thirty increased [respectively] by nine, six, one, and twenty-four, fifteen, and five are the sine-differences for the arcs of 15-degree intervals (half-signs).

- The last quarter of the above verse (highlighted) presents how to find the intermediate values.

Abbreviating sine tables: Brahmagupta's *Khaṇḍakhādyaka*

त्रिंशत्सनवरसेन्दुर्जिनतिथिविषया

Thirty increased [respectively] by nine, six, one, and twenty-four, fifteen, and five

arc	sine	sine diff.
0	0	0
15	39	39
30	75	36
45	106	31
60	130	24
75	145	15
90	150	5

- *Khaṇḍakhādyakā* is a *karaṇa* text — its goal is simplification
- Only six values!
- Arc-multiples of 15°
- Small Radius: $R = 150$
- Interpolation algorithm to find intermediary values

From verse to graphical table

A scribe spontaneously draws a table alongside the verse in a manuscript:

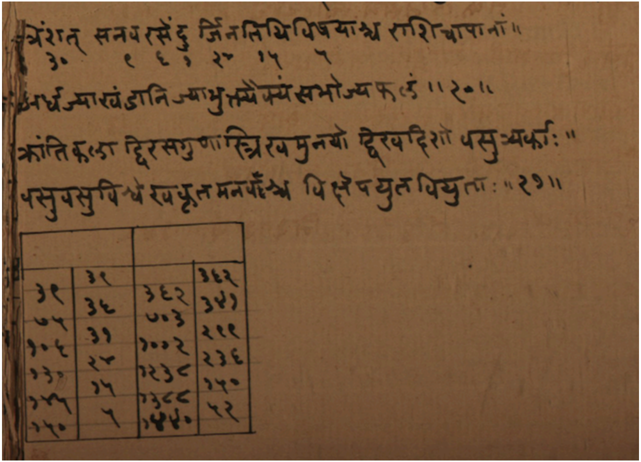


Figure: A manuscript of Brahmagupta's *Khaṇḍakhādyaka* (Anandasrama Institute in Pune)

Enter Āmarāja
a Sanskrit commentator
almost 500 years later

त्रिशत्खनवरसेन्दु-
जिनतिथिविषया।३८।३६।३१।२४।१५।५।ग्रहार्द्धचापानाम्
षड्ज्याखण्डानि
ज्याभुक्त्यैक्यं सभोग्यफलम् ॥८॥

तत्र च रसाश्च इन्दुश्च नवरसेन्दवस्तैः सह वर्त्तन्ते एवंविधाः पृथक् त्रिंशत् जिनतिथिविषयाश्च एतानि षट् राश्यर्द्धस्य खण्डकानि ।

तत्र प्रथमराश्यर्द्धस्य सनवत्रिंशदेकोनचत्वारिंशत् अर्द्धज्या। द्वितीय-राश्यर्द्धस्य सरसत्रिंशत् षट्त्रिंशत् * "तृतीयराश्यर्द्धस्य सेन्दुत्रिंशत् एकत्रिंशत्। चतुर्थराश्यर्द्धस्य जिनाश्चतुर्विंशतिः। पञ्चमराश्यर्द्धस्य तिथयः पञ्चदश। ष"ष्ठस्य विषयाः पञ्च। एवं षण्णां चापार्द्धानां षडर्द्धज्या भवन्ति। एतैः षड्भिः खण्डकैः भुक्तफलैक्यं भोग्यफल-सहितमिष्टा ज्या भवति। अयमर्थः। भुजीकृतराश्यादिकं कला-पिण्डं विधाय नवशत्या भजेल्लब्ध * भोग्यखण्डकफलं भवति। तद्भुक्तखण्डकैक्येनयुतमिष्टज्या भवति।

अत्र वासना। तत्र तावत् खण्डकोत्पत्तिः। सलिलसमी-कृतायामवनौ व्यासार्द्धाङ्गुलप्रमाणकर्कटकेन वृत्तमभिलिख्य दिग-ङ्कितं द्वादशभागाङ्कितं च विदध्यात्। तस्य षडंशो भागद्वयम्। ततो दिग्रेखां मध्ये विधाय तदुभयाग्रसक्तं सूत्रं दक्षिणोत्तरायतं प्रागपरायतं च ज्यासंस्थानार्थं प्रसार्यम्। तदवश्यं व्यासार्द्धतुल्यं जायते। तस्यार्द्धं दक्षिणोत्तररेखावच्छिन्नं प्रागपररेखावच्छिन्नं च राशिज्या तदेवं सकला त्रिराशिज्योच्यते। तच्च व्यासार्द्धमुच्यते। तथात्र तत्कल्पितम् १५० अथास्मादन्येषामुत्पत्तिः। अस्यदलं ७५ इयमेकराशिज्या। † अस्या वर्गः ५६२५। 4"त्रिज्यावर्गः २२५००" अनयोरन्तरम् १६८७५ अस्मान्मूलं द्विराशिज्या १३०। एषा चतुर्थी ज्या अस्यास्त्रिज्यायाश्च विवरम् २० एष शरः प्रथमज्यायां जायते। त्रिज्याकृतिः २२५०० अस्या दलम् ११२५० अतो

मूलम् १०६ इयं तृतीया ज्या । अथ शरकृतिः ४०० राशिज्याकृतिः ५६२५ अनयोर्युतिः ६०२५ अस्या मूलम् ७८ अस्य दलम् ३९ इयं प्रथमज्या । अस्यावर्गः १५२१ त्रिज्यावर्गः २२५०० अनयोर्विशेषः २०९७९ अस्मान्मूलं पञ्चमी ज्या १४५ षष्ठी ज्या त्रिज्यैव । एताः क्रमेणज्याः ७५, १०६, १३०, १४५, १५० एता अधो विशुद्धा जातानि यथा पठितानि खण्डानि ३९, ३६, ३१, २४, १५, ५ ।

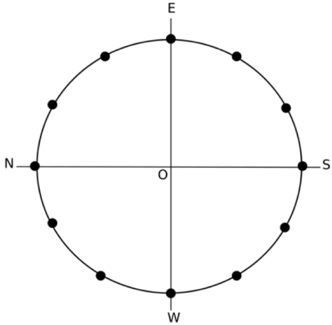
अथैषामुत्पत्तिः । परिधेः षड्भागस्य या ज्या सा तावद्व्यासार्द्धतुल्यैव जायते । ततो द्विराशिज्याया अर्द्धमवश्यमेकराशिज्यार्द्धं भवति स भुजः प्रकल्प्यस्ततोऽस्य व्यासार्द्धं कर्णस्तयोर्वर्गान्तरमूलं कोटिर्जायते । सा तत्पदशेषस्य राशिद्वयप्रमाणस्य ज्यार्द्धरूपतयाऽवतिष्ठते ताञ्च कोटिव्यासार्द्धादपास्य शेषं शरः प्रथमज्याया भुजरूपायाः कोटिरूपः । ततो भुजकोटिकृतियुतिपदं कर्णः । स च राशिचापखण्डस्योभयाग्रसंस्पर्शी ज्यारूपोऽवतिष्ठते । तस्यार्द्धमर्द्धराशिज्या भवति । तद्वर्गं व्यासार्द्धवर्गाद्विशोध्य शेषस्य पदमर्द्धराशिज्याखण्डस्य भुजरूपस्य कोटिर्भवति । सा च तच्छेषस्य सार्द्धराशिद्वयस्यार्द्धज्या भवति । सार्द्धराशिचापं च पदार्द्धेऽवस्थितम् । पदं राशित्रयप्रमाणम् । तस्य चार्द्धयोस्तुल्यत्वाद्भुजकोटी समे स्याताम् । ततः समचतुरस्रक्षेत्रं जायते । तत्रापि व्यासार्द्धं कर्णः । समभुजकोट्योर्वर्गयुतितुल्यः कर्णवर्गः । तस्यार्द्धं भुजवर्गः कोटिवर्गो वा तुल्यत्वादवशिष्यते । तत्पदं भुजः कोटिर्वा । तदेव सार्द्धराशिज्यार्द्धं भवति । पदस्य तु राशित्रयप्रमाणस्य ज्यार्द्धं व्यासार्द्धमेवेति । खण्डकानां च राश्यर्द्धान्निबद्धत्वान्नवशत्या खण्डकलब्धिर्भवति । स्पष्टखण्डकाञ्च फलानयने त्रैराशिकम् । यदि नवशत्या कलाभिः स्पष्टखण्डकं लभ्यते तद्विकलेन किमिति । इति सर्वमुपपन्नम् ॥

Following Āmarāja: Building Brahmagupta's Sine table

- Āmarāja provides a insight into the actual practices of construction that practitioners used to visually demonstrate the process.
- The tools that are required for performing the experiment are the following:
  - Ropes (रज्जु / गुण)
  - Nails / Wooden pegs (शङ्कु)
  - Flat piece of earth (समीकृतावनि)
  - Compass (कर्कटक) to mark the arcs
- How does one ensure a flat piece of earth?
- By using clay and stagnating water.
- What is the compass for?
- To divide the circumference into exactly 12 parts.

Building Brahmagupta's Sine table R = 150

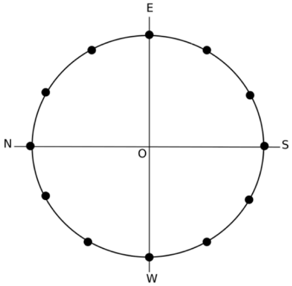
अत्र वासना – तत्र तावत् खण्डकोत्पत्तिः । सलिल-समीकृतायामवनौ व्यासार्द्ध-अङ्गुल-प्रमाणकर्कटकेन वृत्तमभिलिख्य दिगङ्कितं द्वादशभागाङ्कितं च विदध्यात् ।



Here is the rationale – as to how the first order sine-differences (khaṇḍaka) get generated. On the ground made level using water, having inscribed a circle with a compass with measure in digits (equal) to the radius, one should produce (the circle) marked with the directions and marked with 12 parts.

Building Brahmagupta's Sine table R = 150

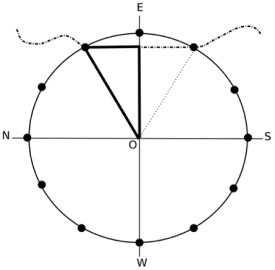
तस्य षडंशो भागद्वयम् । ततो दिग्रेखां मध्ये विधाय तदुभयाग्रसक्तं सूत्रं दक्षिणोत्तरायतं प्रागपरायतं च ज्यासंस्थानार्थं प्रसार्यम् । तदवश्यं व्यासार्धतुल्यं जायते ।



Two parts are a sixth of that (i.e., the circumference). Then, having produced in the middle (of the circle) a direction-line (for the appropriate cardinal directions), a string, attached to both tips of this (arc produced from 2 parts of the circle), extending south-and-north or extending east-and-west, is stretched out for the sake of establishing the chord. This (string) is necessarily equal to the radius.

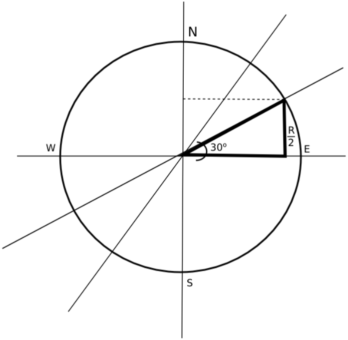
Building Brahmagupta's Sine table R $= 150$

तस्यार्धं दक्षिणोत्तररेखावच्छिन्नं प्रागपररेखावरच्छिन्नं च राशिज्या । तदेवं सकला त्रिराशिज्योच्यते । तच्च व्यासार्धमुच्यते । तथात्र तत्कल्पितम् १५० । अथास्मात् अन्येषामुत्पत्तिः । अस्य दलं ७५ । इयमेकराशिज्या ।



Half of this (string) split by the south-north line or split by the east-west line is the Sine of one sign. This being the case, the Sine of three signs is said to be the whole of it (i.e., the string). And this was said to be the radius. And here it is chosen to be 150. It is from this the other Sines are generated. Half of that is 75. This is the sine of one rāśi.

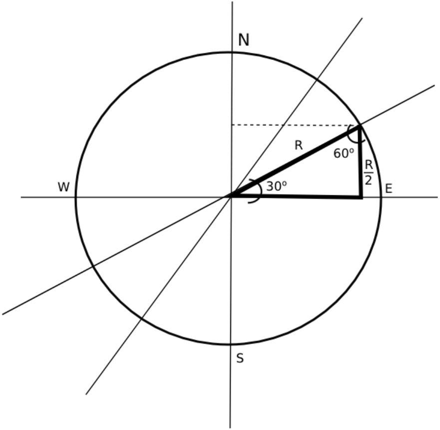
Building Brahmagupta's Sine table R = 150



Half of that is 75. This is the sine of one *rāśi*.

$$\begin{aligned} \text{Sin}30 &= \frac{150}{2} \\ &= 75 \end{aligned}$$

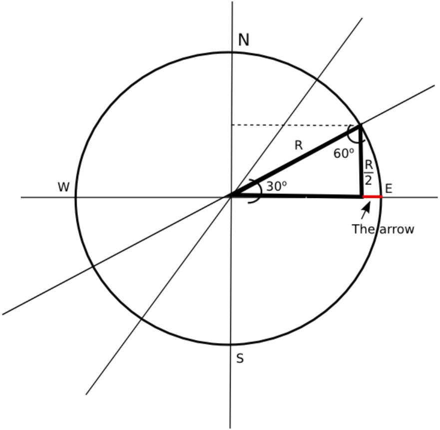
Building Brahmagupta's Sine table R = 150



The square of that is 5625. The square of the Radius is 22500. The difference of the two is 16875. The square root from that is the sine of two *rāśis*: 130. This is the fourth sine.

$$
\begin{aligned}
\mathrm{Sin}60 &= \sqrt{R^2 - (75)^2} \\
&= \sqrt{22500 - 5625} \\
&= \sqrt{16875} \\
&= 130
\end{aligned}
$$

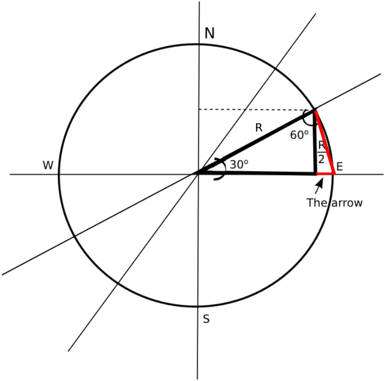
Building Brahmagupta's Sine table R = 150



The difference of this and the Radius is 20. This 'arrow' (*śara*) is produced from the first sine.

$$R - 130 = 20$$

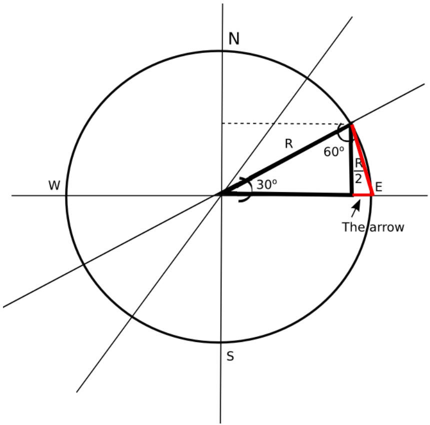
Building Brahmagupta's Sine table R = 150



Now the square of the 'arrow' is 400. The square of the sine of a *rāśi* is 5625 The sum of the these two is 6025. The square root of this is 78. The half of this is 39. This is the first sine.

$$
\begin{aligned}
\text{Sin}15 &= \tfrac{1}{2}\sqrt{(20)^2 + (75)^2} \\
&= \tfrac{1}{2}\sqrt{6025} \\
&= \tfrac{1}{2}78 \\
&= 39
\end{aligned}
$$

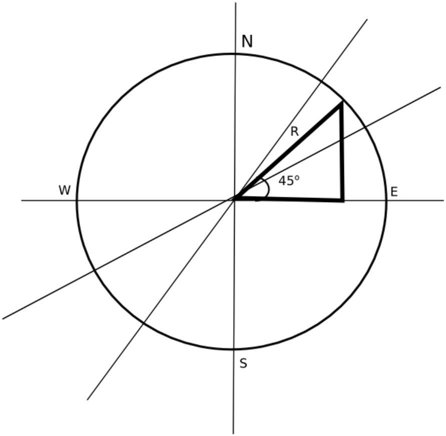
Building Brahmagupta's Sine table $R = 150$



The square of this is 1521. The square of the radius is 22500. The difference of the two is 20979. The square root from that is the fifth sine, 145.

$$\begin{aligned} \text{Sin}75 &= \sqrt{(150)^2 - (39)^2} \\ &= 145 \end{aligned}$$

Building Brahmagupta's Sine table R = 150



The square of the Radius is 22500. Half of this is 11250. Then the square root is 106. This is the third sine.

$$\begin{aligned} \mathrm{Sin}45 &= \sqrt{\frac{R^2}{2}} \\ &= \sqrt{\frac{22500}{2}} \\ &= 106 \end{aligned}$$

Building Brahmagupta's Sine table R = 150

The sixth sine is thus the Radius. These are the sines in order: 39, 75, 106, 130, 145, 150. These are subtracted below. For instance the blocks which have been recited (in the verse) are: 39, 36, 31, 24, 15, 5.

Determining sine values not given in the table

Interpolation algorithms to compute non-tabulated values — the linear approach:

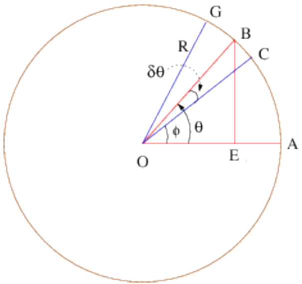
> *... [A desired (i.e., non-tabulated)] sine is the sum of the previous (bhukta) sine (first-differences) with the proportional part (phalam) of the current (bhogya) [sine first-difference].*

θ	Sinθ	ΔSinθ
15	39	39
30	75	36
45	106	31
60	130	24
75	145	15
90	150	5

$$\begin{aligned} R\sin 24 &= 39 + \frac{36 \times (24 - 15)}{15} \\ &= 60.6 \end{aligned}$$

Higher-order interpolation in *Khaṇḍakādyaka* (665 CE)

गतभोग्यखण्डकान्तर-
दलविकलवधात् शतैर्नवभिराप्त्या ।
तद्युतिदलं युतोनं
भोग्यादूनाधिकं भोग्यम् ।। १७ ।।



From the product of the residue (vikala) and half of the difference of the current first-difference (bhogyakhaṇḍa) and the previous [first-difference (gata)], divided by nine hundred, half the sum of that (i.e., the current and the previous first-difference) is to be increased or decreased [depending on whether] the current first-difference is smaller or greater. [The result is the more accurate] current first-difference.

Non-tabulated sines: the higher-order approach

Brahmagupta gives a second order interpolation formula, essentially equivalent to Stirling's finite difference formula, a modern interpolation technique:

$$\mathrm{Sin}\theta \approx \mathrm{Sin}_{n-1} + \frac{\theta - 900(n-1)}{900} \cdot \left(\frac{\Delta\mathrm{Sin}_{n-1} - \Delta\mathrm{Sin}_{n}}{2} \pm \frac{\Delta\mathrm{Sin}_{n-1} - \Delta\mathrm{Sin}_{n}}{2} \cdot \frac{\theta - 900(n-1)}{900} \right)$$

so that:

$$\begin{aligned} R\sin 24 &= 39 + \frac{9 \times 60}{15 \times 60} \left(\frac{39+36}{2} - \frac{9 \times 60}{15 \times 60} \times \frac{39-36}{2} \right) \\ &= \ldots \\ &= 39 + 21.96 \\ &= 60.96 \end{aligned}$$

NOTE: Compare this value with the modern value 61.0105

Approximating sine: Bhāskara I's algebraic algorithm

- Method for computing the sine which avoids geometry
- *Mahābhāskarīya* (ca. 600 CE) 'Great (work) of Bhāskara'
- Rational approximation to the sine, a ratio of two quadratic functions

$$R\sin\theta = R\cdot\frac{4\theta(180-\theta)}{40500-\theta(180-\theta)}$$

- Used by many subsequent authors to avoid having to use trigonometry altogether in their astronomical algorithms.
- Different methods have been proposed by scholars as to how this could have been derived.
- We present one of them that is quite interesting and instructive.

The accuracy of this algebraic approximation

मख्यादिरहितं कर्म कथ्यते तत्समासतः । चक्रार्धांशकसमूहात् विशोध्या ये भुजांशकाः ॥
तच्छेशाषगुणिता द्विष्ठाः शोध्याः खखेषुखाब्धितः । चतुर्थांशेन शेषस्य द्विष्ठमन्त्यफलाहतम् ॥

$$R\sin\theta = R\cdot\frac{4\times\theta(180-\theta)}{40500-\theta(180-\theta)}$$

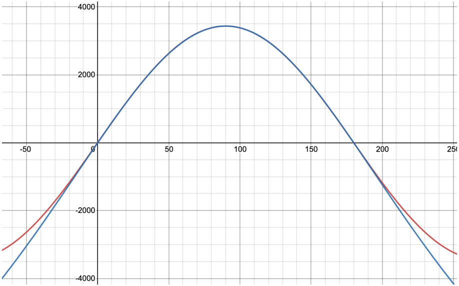


Figure: The sine function (red) and Bhāskara I's rational approximation (blue).

Properties mirrored by Bhāskara's approximation

- We know that sine function
  1. is symmetric about 90° point
  2. is concave over the range $0^\circ \to 180^\circ$
- The formula given by Bhāskara clearly satisfies these properties

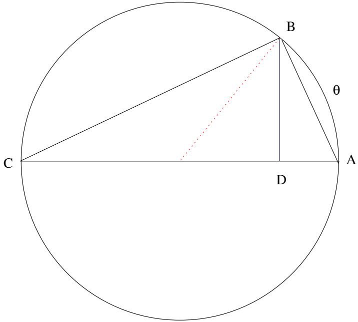
Bhāskara's approximation

$$\sin\theta = \frac{\mathbf{4}\theta(\mathbf{180}-\theta)}{\mathbf{40500}-\theta(\mathbf{180}-\theta)}$$

- Isn't a mathematician's delight to arrive at an expression for sine function that at once captures the properties as well as serves as a very good approximation ($>$ 99% accuracy) for the entire range ($0 - 180^\circ$)?
- Here I may quote the statement made by Hardy[3]
  The Greeks were the first mathematicians who are still 'real' to us today. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. ...

[3] G. H. Hardy, *A Mathematician's Apology* Cambridge, 2nd ed. (1967) p. 80.

Proof of Bhāskara's formula



In the figure, the area of the triangle ABC can be expressed in two ways:

$$A = \frac{1}{2}AB.BC = \frac{1}{2}AC.BD$$

$$\text{or} \quad \frac{1}{BD} = \frac{AC}{AB.BC} \qquad (5)$$

Since the length of the chord $<$ that of the arc, (9) may be expressed as an inequality

$$\begin{aligned} \frac{1}{BD} &> \frac{AC}{\overset{\frown}{AB}\,.\,\overset{\frown}{BC}} \\ \text{or} \quad \frac{1}{BD} &= \frac{x.AC}{\overset{\frown}{AB}\,.\,\overset{\frown}{BC}} + y \\ &= \frac{2xR}{\theta(180-\theta)} + y \\ \text{or} \quad R\sin\theta &= \frac{\theta(180-\theta)}{2xR + \theta(180-\theta)y} \qquad (6) \end{aligned}$$

Proof of Bhāskara's formula

Substituting $\theta = 30$ and $\theta = 90^\circ$ in (10) we have,

$$2xR + 4500y = \frac{9000}{R} \tag{7}$$

$$2xR + 8100y = \frac{8100}{R} \tag{8}$$

Solving the above equations for x and y we have,

$$y = -\frac{1}{4R} \qquad \text{and} \qquad 2xR = \frac{40500}{4R} \tag{9}$$

Using the above values in (10), we have

$$R \sin\theta = \frac{4\theta(180-\theta)R}{40500 - \theta(180-\theta)} \tag{10}$$

which is the same expression given by Bhāskara.[4]

[4] The above proof has been given by K. S. Shukla in his edition of the text *Mahābhāskarīya* with translation and annotation.

Computing Sine tables:

Nityānanda's

Sarvasiddhāntarāja

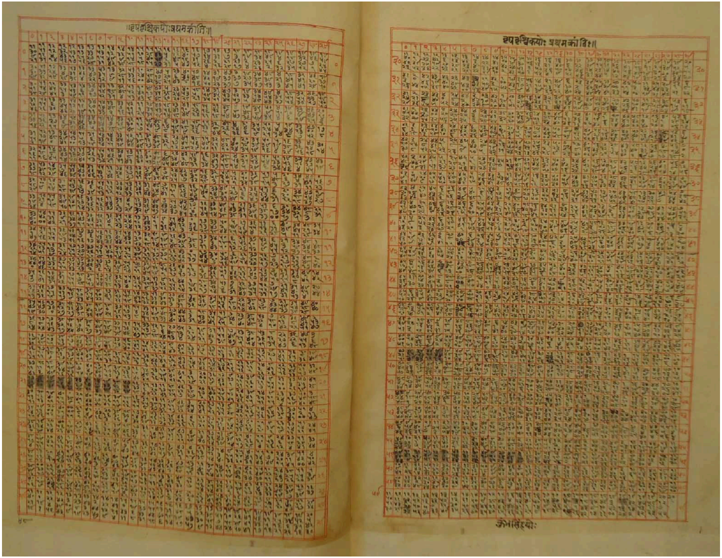
1639 CE

Works of Nityānanda

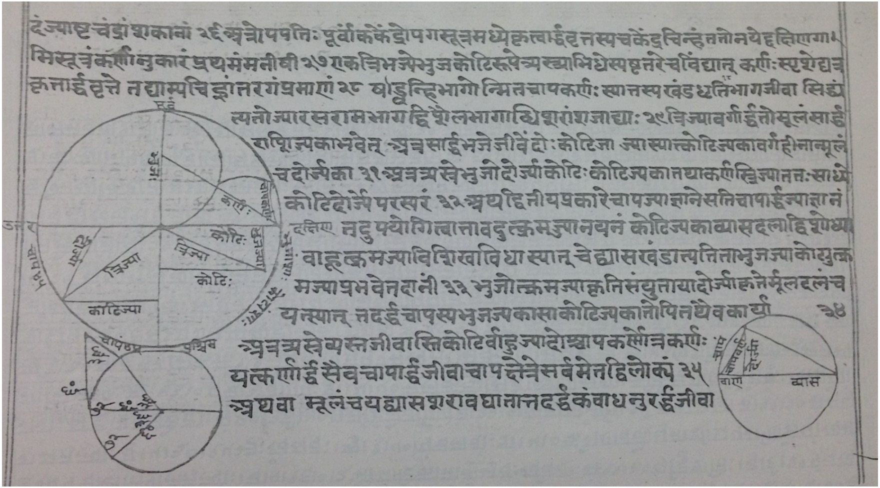
- *Sarvasiddhāntarāja* is a monumental treatise on astronomy composed in beautiful verses by Nityānanda (ca. 1639 CE)
- This work both in its style of composition and its technical content is comparable to the *Siddhāntaśiromaṇi* of Bhāskarācarya (12th century)
- Astronomer at the court of Shāh Jahān
- Well known for tabular works
  - *Amṛtalahāri*
  - *Siddhāntasindhu*

The *Amṛtalahārī*

The *Siddhāntasindhu* (epoch 1628)



The trigonometry section in the *Sarvasiddhāntarāja*



Computing the sine of 90°, 30°, and 18°

नवत्यंशजीवा भवेद्व्यासखण्डं
तदर्धं खरामांशजीवा निरुक्ता ।
तदर्धोनितं विस्तृतेरङ्घ्रिवर्गात्
सपादात् पदं ज्याष्टचन्द्रांशकानाम् ॥ २४ ॥

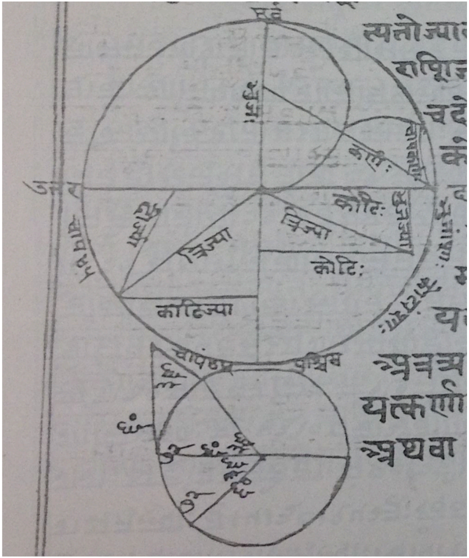
The Sine of ninety degrees is the Radius. The Sine of thirty (*kha-rāma*) degrees is stated to be half of that. The Sine of eighteen degrees is the square-root of the square of quarter of the Diameter increased by quarter [of that amount], decreased by half of that (i.e., half of quarter the diameter, that is, a quarter of the Radius).

$$\mathrm{Sin}\,90 = R,$$

$$\mathrm{Sin}\,30 = \frac{R}{2},$$

$$\mathrm{Sin}\,18 = \sqrt{\left(\frac{D}{4}\right)^2 + \frac{1}{4}\left(\frac{D}{4}\right)^2} - \frac{1}{2}\cdot\frac{D}{4}.$$

Computing the sine of 90°, 30°, and 18°



Verses 40–48: Sine of the sum (and difference) of two arcs

अन्योन्यकोटिमौर्व्या गुणिते ये चेष्टचापयोर्दोर्ज्ये ।
त्रिज्योद्धृते तयोर्यः योगः सा चापयोगज्या ॥ ४१ ॥
anyonyakoṭimaurvyā guṇite ye ceṣṭacāpayor dorjye |
trijyoddhṛte tayor yaḥ yogaḥ sā cāpayogajyā || 41 ||
The Sines of the two desired arcs are mutually multiplied by the Cosines of the other two; when divided by the Radius, whatever is sum of the two, that is the Sine of the sum of the [two] arcs.

अन्योन्यकोटिमौर्व्या गुणिते ये चेष्टचापयोर्दोर्ज्ये ।
त्रिज्योद्धृते तयोर्या वियुतिः सा चापविवरज्या ॥ ४९ ॥
anyonyakoṭimaurvyā guṇite ye ceṣṭacāpayor dorjye |
trijyoddhṛte tayor yā viyutiḥ sā cāpavivarajyā || 49 ||
The Sines of the two desired arcs are mutually multiplied by the Cosines of the other arcs; when divided by the Radius, whatever is the difference of the two, that is the Sine of the difference of the [two] arcs.

Verses 40–48: Sine of the sum (and difference) of two arcs

अन्योन्यकोटिमौर्व्या गुणिते ये चेष्टचापयोर्दोर्ज्ये ।
त्रिज्योद्धृते तयोर्यः योगः सा चापयोगज्या ॥ ४१ ॥

anyonyakoṭimaurvyā guṇite ye ceṣṭacāpayor dorjye |
trijyoddhṛte tayor yaḥ yogaḥ sā cāpayogajyā || 41 ||

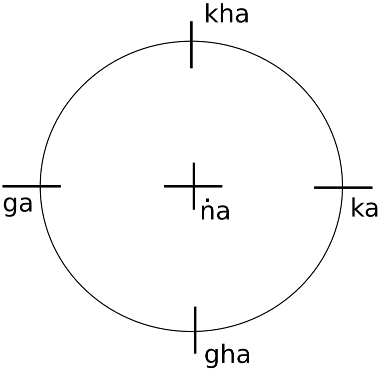
The Sines of the two desired arcs are mutually multiplied by the Cosines of the other two; when divided by the Radius, whatever is sum of the two, that is the Sine of the sum of the [two] arcs.

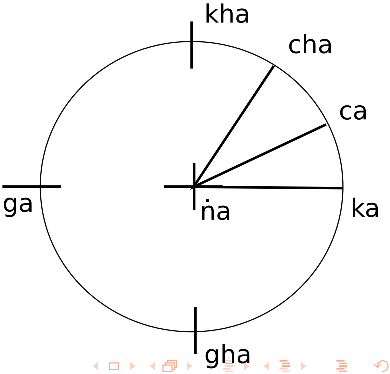
$$\mathrm{Sin}(\theta+\phi)=\frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}+\frac{\mathrm{Cos}\,\theta\,\mathrm{Sin}\,\phi}{R}. \tag{11}$$

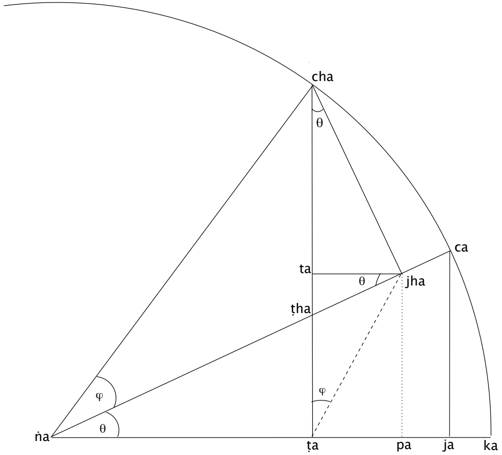
Geometrical demonstration

कखागघं भूमितलेषु मण्डलं ङकेन्द्रकं कर्कटकेन साधयेत् ।
कचं चछं चापयुगं कङं चङं छङं क्रमाद्व्यासदलत्रयं लिखेत् ॥ ४२ ॥

May one draw a circle [labeled] *ka*, *kha*, *ga*, *gha* with a compass on a flat surface, with center [labeled] *ṅa*. [On that circle], may one mark two arcs [with pairs of letters] *ka*-*ca* and *ca*-*cha*. [Then], sequentially, may one draw the three radii *ka*-*ṅa*, *ca*-*ṅa* and *cha*-*ṅa*.







ca-ja = *jha-ṭa*	$\mathrm{Sin}\,\theta$	*ṅa-ja*	$\mathrm{Cos}\,\theta$
cha-jha	$\mathrm{Sin}\,\phi$	*jha-ṅa*	$\mathrm{Cos}\,\phi$
cha-ṅa = *ṅa-ca*	R	*cha-ṭa*	$\mathrm{Sin}(\theta+\phi)$
jha-pa = *ta-ṭa*	$\frac{\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\phi}{R}$	*jha-ta*	$\frac{\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\phi}{R}$

Concluding remarks

- It is quite interesting to know the evolution of different techniques in India to accurately evaluate Rsines – which is ubiquitous !
- The efforts taken by Indian astronomers and mathematicians can be seen in:
  - improving the accuracy of the sine table (which forms the basis for planetary computations)
  - finding various techniques for determing $\sin(A \pm B)$
  - in obtaining a good rational approximation
  - in getting away with sine function calculation (Gaṇeśa)
  - in arriving at the infinite series.
- The absolute logical rigor with which the results have been arrived at is indeed remarkable. Why were they concerned about very accurate values of sines ?
- Accuracy of *Trijyā* R $\rightarrow$ Accuracy in the computation of sines $\rightarrow$ Accuracy in planetary positions $\rightarrow$ Accuracy in the determination of *tithis*, and so on, $\rightarrow$ Avoid incompleteness.[5]

[5]नास्ते कालावयवकलना ...श्रौतस्मार्तव्यवहृतिरपि छिद्यते तत्र धर्माः।

THANK-YOU!