IIT GANDHINAGAR

History of Mathematics in India (HoMI) Project

YOUNG SCHOLARS' CONFERENCE

on

HISTORY OF MATHEMATICS

17 & 18 April 2021

BOOK OF ABSTRACTS

Conference Programme

Notes: 1. Every presentation of 20 minutes will be followed by a discussion of 10 minutes with our experts and audience.
2. All timings are in Indian Standard Time.

Day 1: 17 April 2021

10:00—10:15 am ***Inauguration***

10:15—10:45 am **K. Mahesh**
Nīlakaṇṭha's commentary on the Gaṇita section of Āryabhaṭīya

10:50—11:20 am **G. Rajarajeswari**
Iterative process for obtaining the Infinite series for the Sine and Cosine functions in the Kerala work, Mahājyānayanaprakāraḥ

11:25—11:55 am **Athira K. Babu**
Algebraic operations of surds in Bījapallava, a unique commentary on Bījagaṇita

12:00—12:30 pm **M. Devanand Mallayya**
Mathematical application of Mādhava's Series approximation and convergence acceleration techniques

Day 2: 18 April 2021

10:00—10:30 am **Keshav Melnad**
A glimpse of astronomical table text through Haridatta's Jagadbhūṣaṇa

10:35—11:05 am **Akhilesh Kumar V. & Shailaja D. Sharma**
A survey of computational methods in Kolam

11:10—11:40 am **Sahana Cidambi**
Reconciling error: Mathematical innovations of Gaṇeśa Daivajña

11:45—12:15 pm **Manosij Ghosh Dastidar & Rohitashwa Sarkar**
A brief history actual infinity: Aristotle, Leibniz, Cantor

12:20—12:40 pm ***Concluding Discussion***

K. Mahesh

Research Scholar, IIT Bombay

Dr. K. Mahesh, a Research Scientist in IIT Bombay, has been working in the area of history of mathematics and astronomy with a focus on source works in Sanskrit. He did his post-graduation (Ācārya in Siddhānta Jyotiṣa) at the Rashtriya Sanskrit Vidyapeetha, Tirupati and then did PhD in IIT Bombay (2010) with Prof. K. Ramasubramanian. He did his post-doctoral research at CNRS & University of Paris. A number of his research papers have been published in peer-reviewed journals. His two books, *Critical edition of Karaṇābharaṇa*, and *Studies in Indian Mathematics and Astronomy* (co-edited), have been published recently. The Indian National Science Academy has conferred the prestigious Young Historian of Science Award on him in 2015 in recognition of his significant research contribution.

Nīlakaṇṭha's commentary on the Gaṇita section of Āryabhaṭīya

The *Āryabhaṭīya* is one of the valuable works on astronomy and mathematics, authored by Āryabhaṭa at the end of 5th century CE, that received favourably throughout India. As an evidence of its reputation, we find several commentaries on it composed by later astronomers. Apart from that we see many independent works and karaṇa texts based on it which were also popular. Among the commentaries on the Āryabhaṭīya that have been authored so far, the *Āryabhaṭīya-bhāṣya* of Nīlakaṇṭha Somayājī (1444–1544 CE) is by far the best and most elaborate one. He presents multi-fold reasoning to the enunciations of Āryabhaṭa along with a number of citations of authority, illustrations and various related topics. Some of its distinct features found in the Gaṇita section of it would be discussed, during the presentation.

G. Rajarajeswari

Research Scholar, Department of Sanskrit, University of Madras

G. Rajarajeswari, is a Research Scholar at University of Madras, Department of Sanskrit. She completed an M.Phil in Indian Astronomy and has worked on *Mahājyānayanaprakāraḥ*, a manuscript on derivation of sine series, under the supervision of Prof. M.S. Sriram, President of the Prof. K.V. Sarma Research Foundation, Chennai. She is currently working on Triprasnadhyaya in Indian astronomy.

Iterative process for obtaining the Infinite series for the Sine and Cosine functions in the Kerala work, *Mahājyānayanaprakāraḥ*

It is well known that the infinite series expansion for the sine and cosine functions were first discussed in the Kerala works on astronomy and mathematics, and are invariably ascribed to Mādhava of Sangramagrama (14th century CE). The full proofs of these are to be found in *Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva (composed around 1530 CE). However there is a Kerala work called *Mahājyānayanaprakāraḥ* which describes the infinte series for the *jyā* (Rsin θ) and the *śarā* (R(1-cos θ)) and provides a shorter derivation of them. This was discussed in a paper by David Gold and David Pingree in 1991. However, that paper did not explain the derivation of the infinite series in the manuscript. In this paper we provide the derivation completely based on the *upapatti* provided by the author of the manuscript himself. This derivation is very similar to the one in *Gaṇita-yukti-bhāṣā* , but differs from it in some respects. Both the derivations are based on the iterative solution of the discrete version of the equations:

$$sin\theta = \int_0^\theta cos\theta' d\theta' = \theta - \int_0^\theta (1 - \cos\theta')d\theta',$$

$$1 - cos\theta = \int_0^\theta sin\theta' d\theta' ,$$

and make use of the result: $\sum_{j=0}^{n-1} j^k \approx \frac{n^{k+1}}{k+1}$, for large n . This result is crucial for obtaining the infinite series for π also. However, no iterative process is involved there.

Athira K. Babu

Research Scholar, Department of Sanskrit Sahitya, Sree Sankaracharya University of Sanskrit, Kerala

> Athira K. Babu is a research scholar at the Department of Sanskrit Sahitya, Sree Sankaracharya University of Sanskrit, Kalady, Kerala, and pursues her PhD on "Rationales in Indian Mathematics: A Study with Special Focus on the Sanskrit Commentaries of Bījagaṇita". Her research mainly focuses on Indian mathematics, the Kerala School of Mathematics, and ancient Indian chemistry. She completed her graduation, BEd and post-graduation in mathematics from Mahatma Gandhi University, Kottayam, Kerala and also a post-graduation in Sanskrit Sahitya from Sree Sanskracharya University of Sanskrit, Kalady, Kerala. She completed an M Phil from the same university on the title "A Review on Veṇvāroha by Mādhava of Saṅgamagrāma".

Algebraic operations of surds in Bījapallava, a unique commentary on Bījagaṇita

The Sanskrit term *karaṇī* is used for surd, the irrational root of an integer. In *Siddhānta Śekhara*, Śrīpati defines: "The number whose square root cannot be obtained exactly is said to form an irrational quantity *karaṇī* (ग्राह्यं न मूरं खरु यस्य राशेस्तस्य प्रततष्ठं करणीतत नाम ।)". It can be seen that the elementary treatments of surds, particularly addition, subtraction, multiplication, separation and extraction of square root of surds, compound of surds and the like occurs in the algebraic works on Indian mathematics. The present article gives an account of the treatment of surds in *Bījapallava*, a unique commentary on *Bījagaṇita*, algebra in Sanskrit by Bhāskarācārya.

The Sanskrit term Gaṇitaśāstra, meaning literally the "science of calculation", is used for mathematics. According to *Vedāṅga Jyotiṣa,* "Like the crests on the heads of peacock, as the gems on the hoods of cobras, so is mathematics at the top of all science." This statement reveals the importance given to mathematics in ancient India. In Sanskrit, algebra is known as *avyaktagaṇita* or *bījagaṇita* and deals with the determination of unknown entities, while arithmetic (*vyaktagaṇita* or *pāṭīgaṇita*) deals with the mathematical operations with known entities.

Bhāskarācārya's *Bījagaṇita*, a Sanskrit classic on algebra, is the second chapter of his monumental treatise on mathematics, *Siddhāntaśiromaṇi*.

Bījapallava is a famous Sanskrit commentary on *Bījagaṇita* by Kṛṣṇa Daivajña (16th century). The name *Bījapallava* is a compound formed by the composition of the words *bīja*, meaning algebra, the science of analytical calculation, and *pallava*, meaning 'sprouts'. *Bījapallava* thus means "the sprout of algebra". In the introduction of the edition of *Bījapallava*, T.V. Radhakrishnan writes, "When the advent of spring was visible by the sprouts on the trees Sri Kṛṣṇa Daivajña realised that the tree of algebra also should have its sprouts. So he wrote this commentary and called it the sprout of algebra or *Bījapallava*, announcing to the people all around that this knowledge also was bound to have a better recognition."

Radhakrishna Sastri, the editor of *Bījapallava,* also stated that in the introduction of the work, the author of the original text gives only the general enunciations in original text (the *mūlagrantha*). A commentary consists of the explanatory statements and demonstrations of the general enunciations. Usually, the demonstrations are merely verifications (by examples) in order to understand the text correctly. Here is the relevance of the study of the commentary on Bījagaṇita.

The other Sanskrit commentaries during the medieval period are: The *Sūryaprakaśa* of Sūryadāsa, *Bījavivaraṇa* (1639 CE) of Vīreśvara, *Śiśubodhana* (1652 CE) of Bhāskara of Rājagiri, *Bījaprabodha* (1687 CE) of Rāmakṛṣṇa, *Vāsanābhāṣya* (before 1725 CE) of Haridāsa, *Bālabodhinī* (1792 CE) of Kṛpārāma.

There are two editions of *Bījapallavam* of Kṛṣṇa Daivajña: (1) by Dattatreya Apte, entitled *Bhāskarīyabījagaṇitam* with the *Vyākhyā Navāṅkura* of Kṛṣṇa, published as ASS 99, Poona, 1930; (2) by T.V. Radhakrishna Sastri, entitled *Bījapallavam* with introduction by T.V. Radhakrishna Sastri, published as Madras GOS 67, TSMS 78, Thanjavur, 1958. The work *Bījapallava of Kṛṣṇa Daivajña: Algebra in Sixteenth-Century India, a critical study* by Sita Sunder Ram in 2012 is a recent study.

Bījapallava is divided into thirteen chapters (*adhyāyas*) which contain the six-fold operation of positive and negative quantities, zero, unknowns and surds (*karaṇī*), the indeterminate equations of the first degree (*kuṭṭaka*), and second degree separately; linear and quadratic equations having more than one unknown; operations with products of several unknowns; a section about the author Bhāskara and his works.

This paper tries to demonstrate the reflections of elementary treatment of surds in the mathematical tradition of India with respect to the commentary *Bījapallava* on *Bījagaṇita,* especially under the background of medieval India.

M. Devanand Mallayya

Pursuing Integrated BSMS, IISER-TVM, Thiruvananthapuram, Kerala

M. Devanand Mallayya was born in Thiruvananthapuram in 2000 as the second son of Mrs. Anitha Mallayya and Dr V Madhukar Mallayya. With an ardent desire to pursue higher studies and research in mathematics, after completing Class XII he joined the Integrated BSMS Program of the Indian Institute of Science Education and Research (IISER–TVM) at Trivandrum in 2018, and is now in his sixth semester with Mathematics Major along with Physics and Data Science as Minors.

Mathematical Application of Mādhava's Series Approximation and Convergence Acceleration Techniques

The *Līlāvatī* is a basic mathematical treatise written by Bhāskara II of 12th century AD. Ever since the composition of this delightful treatise, it continued to attract the attention of various scholars from all over India and abroad and as a result a large number of commentaries were written on this basic treatise. Among them, the 16th century commentary *Kriyākramakarī* is found to be the most elaborate and extensive one. It throws a lot of light on various advanced mathematical concepts developed during that period. The first part of the commentary up to the verse 199 of the *Līlāvatī* was written by the famous Kerala mathematician Tṛkuṭṭaveli Śaṅkara Vāriyar (1500 – 1560 AD) and the remaining portion of the commentary was completed by another scholar Mahiṣamangalam Nārāyaṇa (1540 – 1610 AD). The commentators have given detailed expositions and rationale of the *Līlāvatī* verses one by one along with several illustrations, different types of proofs and other related materials from various works of earlier and contemporary mathematicians to update the knowledge in the field. Several new findings and own contributions are also included in the commentary. Some of the sections are supplemented with new materials to enrich the knowledge and enhance the quality of the commentary. In particular the commentary throws a lot of information on several remarkable achievements made by the mathematicians belonging to the Kerala School.

The Kerala School is found to have played an important role in the history of astronomy and mathematics in India and has a long tradition beginning from at least the 4th century AD. But with the emergence of Saṅgamagrāma Mādhava (1340 – 1425 AD) there was a great upsurge of astronomical and mathematical activities and the period from 14th century AD to 18th century AD may rightly be called the golden period in the history of astronomy and mathematics in India. Several remarkable achievements were made by Kerala Scholars belonging to Mādhava school during this period. The extent of mathematical achievement during this period is quite startling and is totally of a different flavor.

Like Āryabhaṭa and others, Kerala mathematicians were also fully aware of the problem of incommensurability of circumference and diameter. But in order to deal with the problem of evaluation of circumference of a circle with desired degree of accuracy Saṅgamagrāma Mādhava made inroads in the field. He broke the barriers of finite and soared into the so far untrodden field of infinite from infinitesimals. Formulating and applying several ingenious concepts and new techniques, Mādhava discovered infinite series expressions for evaluating the circumference of a circle with given diameter and made startling contributions in the highly fertile area of infinite series. Mādhava has also discovered power series expressions for Indian trigonometric functions. Disciples of Mādhava and later scholars belonging to the Mādhava school travelled far and wide along the highway to infinity thrown open by Mādhava and made their own contributions in this field. Quoting Mādhava they discussed several important results relating to infinite series in their works such as the *Tantrasaṅgraha* of Nīlakanṭha Somayāji (1444 – 1545 AD), the *Kriyākramakarī* of Śaṅkara (1500 -1560 AD) and Narayana (1540 – 1610 AD), the *Yuktibhāṣā* of Jyeṣṭhadeva (1500 – 1610 AD), the *Karanapaddhati* of Putumana Somayāji (1660 – 1740 AD) and the *Sadratnamāla* of Śaṅkara Varman (1800 – 1838 AD).

In the verse 199 of the *Līlāvatī* the author Bhāskarācārya has given a rule for finding the value of circumference of a circle from its diameter. While commenting on this rule in the *Kriyākramakarī* the commentator Śaṅkara Vāriyar gives an excellent exposition of the rule along with several citations from the works of earlier and contemporary mathematicians. Thus referring to Mādhava, Śaṅkara gives the series

$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots + (-1)^{n-1}\frac{4d}{2n-1} + (-1)^n 4d\{G_n\}$ where $G_n = \frac{(2n/2)}{(2n)^2+1}$ is the correction factor after n terms for evaluation of circumference C given the diameter d. Without the remainder term after n terms the series is infinite in nature and so by increasing the number of terms better and better approximations can be obtained. But because of slow convergence nature of this infinite series it is clear that one will have to take a very large number of terms to get a desired degree of accuracy. So for practical purpose Mādhava introduced an innovative idea of attaching a remainder term after taking a desired number of terms say n. For a given n the value of C computed from the series with the remainder term is found to give better approximation than the value computed from the series without the remainder term. Śaṅkara Vāriyar has

provided a detailed exposition along with rationale of the method for finding the correction factor (*guṇa*, the multiplier of $(-1)^n 4d$) in the remainder term). Śaṅkara then proceeds to modify the correction factor to reduce the error and consequently increase the degree of accuracy in the computed value of C. Śaṅkara obtains three correction factors G_n', G_n and G_n'' for more accurate correction (*sūkṣmatara saṁskāraḥ*) namely $G_n' = \frac{1}{4n}$, $G_n = \frac{(2n/2)}{(2n)^2+1}$ and $G_n'' = \frac{n^2+1}{[4(n^2+1)+1]n}$. These correction factors are found to be the first three successive convergents of the D.T Whiteside's continued fraction $\frac{1/2}{2n+}\frac{1^2}{2n+}\frac{2^2}{2n+\cdots}$ .The error (*sthaulya*) corresponding to each of these correction factors are also computed from $\epsilon_n = |C_{n+1} - C_n|$ and analyzed. New series deduced from ϵ_n is found to converge rapidly to C. Another great scholar Jyeṣṭhadeva of Mādhava school belonging to 16th century AD has also given a detailed derivation and analysis of infinite series for circumference and series approximation in his famous work *Yuktibhāṣā*.
In both the works namely the *Kriyākramakarī* and the *Yuktibhāṣā* the authors profusely refer to Mādhava and apply several ingenious techniques to approximate, increase accuracy, and accelerate the convergence of infinite series for evaluation of circumference.

Drawing inspiration from the Mādhava techniques, an attempt is made in this paper to use these ingenious techniques to examine the case of various other infinite series (apart from series for π) and find the corresponding correction factors, their continued fraction patterns and error functions. Attempt is also made to accelerate the speed of convergence by extracting new series from the error functions derived from the mother series.

References

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2. *Yuktibhāṣā* of Jyeṣṭhadeva, Part I: Ed. with Notes by Ramavarma Maru Thampuran and A.R Akhileswara Aiyer, Mangalodayam Press, Trichur, 1948, (chapter VI)
3. *Gaṇita-Yuktibhāṣā* of Jyeṣṭhadeva: Part I: Malayalam Text Critically Ed with English Translation by K.V. Sarma with Explanatory Notes by K. Ramasubramanian, M.D Srinivas and M.S Sriram, Hindustan Book Agency, New Delhi, 2008, (chapter VI)

Keshav Melnad

Post Doctoral Fellow, IIT Gandhinagar

From the age of ten, Keshav Melnad underwent the traditional *gurukula* system of education for twelve years, which included learning Vedas, Śāstras, Yoga, fine arts as well as modern science. During this period, he also graduated with two Bachelor's degrees and a Master's degree. Later, having been awarded a JRF, he completed his research in the field of History of Indian Astronomy and Mathematics from IIT Bombay and was awarded a doctoral degree for his thesis, "A Critical Study of Haridatta's *Jagadbhūṣaṇa*". Currently, Keshav works as a Post-Doctoral Fellow in the project "History of Mathematics of India" at IIT Gandhinagar, Gujarat.

A glimpse of astronomical table text through Haridatta's Jagadbhūṣaṇa

Tracing the development of Indian astronomy throws valuable insights on the sociocultural and intellectual interactions with other cultures, as can be seen in the astronomical literature of Siddhāntas, Karaṇas and Tantras, and particularly Koṣṭhakas / Sāraṇīs (tables), which are the primary focus of this talk.

Ancient astronomical tables are perceived as a continuous descent with modification and advancement, spreading through the Babylonian, Greek, and Islamic traditions successively, eventually leading to the manifold tables of modern astronomy in its pre-computer period. The origin of the Indian astronomical tables appears to be collaterally related to this lineage, but with a substantial amount of independent development. Their major contents include mean and true motions of the luminaries, synodic phenomena, eclipses, trigonometry along with necessary parameters for computation. At the times when algebraic notation and mathematical graphing techniques were absent, the tabular format was considered the best way to transmit precise information to the user. As far as the Indian tradition goes, one of the main advantage of such tables would be to achieve significant simplification in the computational process; they became the standard supplement for astronomers as well as astrologers, who through their practice helped compose additional calendrical, astrological texts like almanacs and ephemerides (*Pañcāṅga*).

This talk focuses on one particular text, the Jagadbhūṣaṇa of Haridatta, which gives a simplified method for calculating planetary positions.

Akhilesh Kumar V.

Graduate student at IIIT, Bangalore

Akhilesh Kumar V, 21, completed his Bachelor's in Physics from Azim Premji University in 2020 and is currently pursuing MSc Digital Society at the International Institute of Information Technology, Bengaluru. His research interests include Applied Mathematics, Human-Computer Interaction, Data Analysis, ICT Regulation and Policy Research, Qualitative and Quantitative Research Analysis. He is currently working as an intern as part of the Math Heritage Initiative at NIAS, Bengaluru, a platform established in 2020 to promote the study of and exploration into math and computational aspects, with a focus on young scholars.

with

Shailaja D Sharma

Professor, NIAS Mathematics Heritage Initiative

A Survey of Computational Methods in Kolam

Women in many households in southern India follow a long-standing tradition of decorating the threshold of their home at dawn every day by drawing stylized geometrical patterns on the floor, using rice powder, powdered soapstone, or a paste of rice in water. These traditional drawings, called *kolams* in Tamil Nadu, seem to have ancient origins. Kolams are enigmatic patterns, and have attracted the attention of social anthropologists and mathematicians alike. Although they are 'performed' without reference to a textbook or handbook of patterns, that is to say, they are drawn from so-called muscle memory, they have intrigued onlookers because of their interesting and complex patterns. Kolams have been studied using different computational and mathematical methods, although they have entered into scholarly literature only since the 1970s. Lindenmeyer language, turtle graphics, array grammars, representation using modular numbers and other computational methods are covered in the present survey. We have undertaken a comprehensive reconstruction of several Kolams using multiple methods. Such a comprehensive and comparative study is missing in the literature, which may be partly why these highly interesting patterns have so far drawn a limited amount of

attention. The present study is restricted to Kolams drawn on a rectangular or slanting grid of dots, with lines going around the dots (*pulli kolam*).

In computer science, a formal language is specified by a grammar, where a grammar consists of a set of symbols, along with rules for producing words and sentences. The production rules are also called 're-writing rules' and replace a given string with another string. Picture languages is a formal language where the starting string is a symbol string and re-writing rules are pictures. Kolams generated by such methods are discussed by Maria Ascher (2002).

An important shortcoming of the above method is the absence of reference to the grid of dots, which is the starting point for the Kolams under discussion. Several methods of generating Kolam patterns overcome this shortcoming. Array grammars deal with two-dimensional arrays of symbols instead of strings of symbols and can thus be defined on a grid of dots. Siromoey et al. (1973,1974) give array rewriting rules wherein, by a suitable choice of primitives, terminal and non-terminal symbols, a given array grammar can be seen to correspond to a defined family of Kolams.

In practice, the womenfolk who draw Kolams lay out the *pulli* array first and then straight lines, threads or circles are drawn connecting or encircling the dots. If labelled dots are generated first with the labels contain the instructions on the 'state' of the dot, then the final *kolam* required can be formed by simply reading the metadata contained in the labels The advantage of this method is that a single grammar with a finite number of instructions can generate an infinite set of *kolam* patterns of different sizes.

Kolams can be analysed as being constructed on a grid of dots, which is either a square grid (*ner pulli*) or an inclined grid or rhombic grid (*nadu pulli*). In certain circumstances, Kolams can be associated with a unique combination of hexadecimal numbers, based on the binary coding of the grid dots. Yet another approach models the structure of the Kolam, by defining comprehensively the relationship between neighbouring dots on the dot array, considering each dot as a vertex and the Kolam as a graph. A tree structure, called N-line, is constructed by joining vertices which have certain well-defined relationships and the Kolam is seen as a medial graph cutting across the N-line. This method is introduced by Yanagisawa et al. (2007) and is used to generate large families of computer generated Kolams.

Nagata (2007) treats Kolam as a pattern emerging over time, through directed and continuous moves, similar to voiced language. Tile coding is a way of setting up the status of 4 edges of a tile containing a dot. I. A set of sequential statuses from the starting edge between two adjacent tiles is called a chain code. Tile and Chain Codes can be used to fully describe a Kolam.

We thus describe a large number of different extant computational methods of describing and generating Kolams and list a few additional approaches not included in the present survey. The Kolams so derived may be finite, extendable or infinite. Generated Kolams include fractals and space filling curves. It is clear that Kolams encode rich mathematical and computational content, which has been only minimally explored. Directions for further research are indicated.

Sahana Cidambi

Research Scholar, University of Canterbury, New Zealand

Sahana Cidambi is a PhD student at the University of Canterbury in New Zealand studying Sanskrit mathematical astronomy under the supervision of Professor Clemency Montelle (University of Canterbury, NZ) and Professor Kim Plofker (Union College, USA). Sahana's research focuses on analyzing the mathematics of select chapters of the *Grahalāghava* (composed in 1520 CE) of Gaṇeśa Daivajña as well as comparing the commentarial styles and pedagogical methods of members of his professional lineage, Mallāri and Viśvanātha, in their accompanying commentaries.

Reconciling Error: Mathematical Innovations of Gaṇeśa Daivajña

Astronomers' basic planetary model was one of uniform circular motion with simple concentric orbits. They observed the effects of what we now attribute to the ellipticity and heliocentricity of planetary orbits, and adjusted this basic model to be an epicycle/eccentric model that relied on trigonometry. However, in 1520 Nandigrām, India, Gaṇeśa Daivajña composed his astronomical handbook the *Grahalāghava*, which famously claimed to reject astronomers' dependence on trigonometry for planetary calculations. Instead, he provided algebraic formulae to approximate trigonometric computations. To account for approximation error, Gaṇeśa devised correctional terms or even additional formulae. In this talk, I focus on Gaṇeśa's unconventional inclusion of two corrective formulae to account for oddities in the orbits of Venus and Mars, and will discuss possible motivations for these verses with the commentaries of two prominent members of Gaṇeśa's professional lineage: Mallāri and Viśvanātha.

Manosij Ghosh Dastidar

Pondicherry University

with

Rohitashwa Sarkar

Center for Studies in Social Sciences, Calcutta

A brief history of actual infinity: Aristotle, Leibniz and Cantor

Section 4 of book 4 of Aristotle's Metaphysics introduces the argument against infinity. It begins with a refutation of some contemporary philosophies of difference, notably that of Heraclitus, who argues for simultaneous being and non-being of the same thing. For thinkers like Heraclitus, a thing can both be and not-be at the same time. Aristotle finds this completely absurd. A thing, for example, a man has a particular definition. While it can have infinite attributes, man's definition pertains to his essence, which is composed of a group of necessary attributes : " man is a two-footed mammal". Once this definition is given, man becomes this particular thing and nothing else. One can counter that the same word can have multiple definitions. Aristotle's response is that as long as the multiplicity is finite, we can use different words for different definitions, thereby maintaining that each word has a fixed definition and essence.

Thus, because each word is defined in terms of its essential attributes, it cannot also be the same as it's refutation, which will precisely not have the same essence. This is true for the same reason that 'man' is not the same as 'white' – even though a man can be white. Man and white as terms have different essences, even though they can be accidental attributes of one another– like a 'white man'. But they do not belong to each other's essence. This is why to say that something simultaneously both is, say, white and not-white, is to mix-up the thing's essence – everything as a result loses its individual being and in a world without the stability of essence, everything becomes everything else. This is where Aristotle locates infinity. He has to dismiss it's actuality because for him all modes of meaning-making and communication breakdown if infinity is real.

The law of non-contradiction is the ground of this entire edifice, the urlogical shield against the chaos of infinity. Infinity cannot be an actual object in this universe because, as is evident from the preceding discussion, the very condition for infinity is violation of non-contradiction (something both is and is not) Infinity is the included middle that splits apart the binaries of Aristotelian identity. Within this paradigm, it thus remains out of reach, ineffable, the point at which the system collapses, but also, it stands to reason, it is the point of its obscure origin.

For us, what this means is that the whole structure quilted by the laws of identity, non-contradiction and excluded middles and that relies principally on the notions of substance and essence for the definition of all objects, produces infinity as the aporetic limit of its thought – the limit that is really ground. The truth of the Aristotelian universe thus turns on its head because infinity is in truth it's absolute reference, its sovereign source that is outside it but sanctioning everything that is happening within. The problem with the Aristotelian system is that it willingly forgets it's origin and true being, a forgetting that engenders it's discourse on identity. Throughout this history, we will try to isolate the epistemic breaks in these formal structures – Aristotle gives way to a scholastic theory of grammar which is nonetheless quite derivative of his own work, then Leibniz creates an actual syncategorematic infinite – the latter is properly of the order of a break or rupture. For us, the question will be, what are the epistemic shifts from a classic Aristotelian understanding that allow for the preparation of conditions that can generate an actual infinity? The history we are tracing will attempt to answer this question through the shifts initiated by Leibniz and Cantor.

For a large part, medieval theologians inherited Aristotle's views on the actual infinite – especially influential was his refutation of Zeno in Book III of Physics. Aristotle had said that Zeno's infinity is not an actual entity because one never reaches it, there is always more to it. It is incompossible with the concept of wholeness, and what is not whole or what is never fully itself can never be anything – it is a logical misnomer. The infinite refers to the whole, but in reality we only get a part of the whole – finitude. At the level of actuality, it is always finite. For medievalists, the main concern for infinity was as a divine attribute – infinity not only as the largest quantity, but as the highest perfection. The real task for the medievalists was to rigorously define an infinite being – unlike the neo-Platonic one-infinite, this was divine being was not outside the realm of beings, rather it was the supreme Being. Moreover, its existence could not be disputed. So while at the level of physical things, infinity was never actual, it was a little hard to say that divine infinity was merely potential. It was a completed whole, a totality and therefore absolutely real. Some, like Duns Scotus, even argued that this infinity was quantitative. His argument is almost proto-Cantorian in its imagination – he accepts that as long as we think of infinity as a process that is always heading towards completion but never reaching it, infinity remains potential. But we could think of an infinitely large quantity as actual if all its parts were available as a complete whole simultaneously. One does have to wait for the future to find the completed infinity if one can think of infinity as already completely given, as a particular quantity. Thus, it is true that long before the Leibnizian infinitesimal we have in Duns Scotus an apprehension of infinity as an actual quantity.

For the large part though, despite the obvious antinomies that concern a notion of 'infinite being', Aristotle's views were gospel: till the dawn of calculus in the seventeenth century, infinity was deemed merely potential. Although God was real, there could no natural or logical object that could really encompass infinity. The middle-ages do not constitute any real break from Aristotelian thought. The revival of Greek learning across Europe in the 15th and 16th centuries lead to a flourishing of mathematical thought – which is why in the century before Leibniz, we find apprehensions of a notions that would rigorously defined in calculus later: Fermat's maxima and minima, Cavalieri's indivisibles. The Leibnizian infinite really radicalizes this entire history, which is what we will presently follow.

The first point of entry for us is the distinction between the categorematic and syncategorematic. Initially a medieval theory of logic, the distinction is utilised expertly by Leibniz to create a major change. This will require some unpacking. Every proposition or judgement has a logical constant, apart from having content. Take the sentence:

> Every base is an alkali. OR
>
> No Christian prays to some pagan.

The terms every (universal affirmative), some (particular) and no (universal negative) are logical constants, whereas 'boy', 'girl', 'christian', 'pagan' are subject/predicate terms that supply content to the propositions. The constants give form to an enunciation, whereas the categoremaric subject/predicate terms supply the matter that is organized by the formal constants. The constants are termed syncategorematic.

The categorematic terms, as subjects or predicates, have meaning on their own, but the syncategorematic terms get meaning contextually – 'if', 'then', 'every', 'some' are terms that only produce meaning through attachment with categorematic terms. In the case of the infinite, for most scholastics, infinity was like a syncategorematic term, whose meaning on its own is indefinite. Normal categorematic terms could not be attributed of God – if that was possible, God would be like all other creatures. It is only infinity as a syncategorematic attribute that can be, in a sense, 'predicated' of the Creator. Now this syncategorematic was accepted as a potential infinite – as we've made clear, with a few notable exceptions, infinity as actual was a contradictory concept for the scholastics. Leibniz takes an entirely heterodox position: the syncategorematic infinite is actual. This means that he is not disputing the description of the infinity – infinity remains a process, an indefinite incomplete entity, but completeness or wholeness is no longer necessary for actuality. It is at this precise juncture that Leibniz makes his intervention – something can be actual without being a unity, the actual is fragmented, partial, not composed of atomic wholes but infinitesmals.

Let us remember the precise nature of Leibniz's opposition to atomism. For Leibniz, nature is a continuous plenum, not a whole composed of separable atoms. Every point of the plenum can be further decomposed, every unity breaks down further, there is always more or less, always an outside. The actual is always a process, never a unity. But the parts of nature never separate. It is like the same thing twisting, turning and expanding or contracting without breaking: everything is topologically homeomorphic. The unity is therefore at the level of homeomorphic continuity. It is all One. But at the same time, internally everything is in permanent flux and becoming. The Leibnizian actual is simply the unity of becoming.

What is crucial for us is how he creates modern mathematical infinity by reconceptualizing actuality/existence. It is truths of existence, as opposed to truths of essence, that are extended on to infinity. Let us recall the famous "Caesar crossed the Rubicon" example. The crossing leads to war, which leads to death, which leads to several other events and so on and the chain never terminates. This series is the syncategorematic infinite. What it does is encompass an entire world. Now we are really getting into the heart of the Leibnizian system: every action encloses an entire world. There are infinite worlds with an infinite number of events. Every world is a point of view, and, to use the more well-known expression, a monad.

Thus, existence/actuality is the unfolding of the syncategorematic infinite. This is how Leibniz makes the syncategorematic actual – and how he creates a new conception of actual infinity that pushes Aristotelianism to its limits. If infinity was the limit-point of the Aristotelian world, if it was simultaneously it's terminal point as well as it's transcendental beginning, here we a system that completely internalizes it. It is a real object in this world. It can be shown in detail, though it would be outside the direct purview of this work, that what makes this possible is his radical rehashing of the Principle of Identity, and the added support of the Law of Continuity, the notion of compossibility and his reading of the analytic a priori. It is their combination that ruptures the received history of Aristotelian logic.

It is once again a rupture at the level of the concept 'actuality' that we will trace after this. Georg Cantor gives us a truly mathematical notion of actuality, defined in terms of cardinality. What makes aleph naught and aleph one actual is that they can, in a sense, be measured as cardinality. By making cardinality the ground for actuality, he creates what is perhaps the first properly mathematical and non-philosophical definition of actuality. What follows is a short historical demonstration of how this idea first developed in Hume and went from Galileo, Leibniz to Cantor:

In *A treatise of Human Nature* Hume writes:

> "I have already observed, that geometry, or the art, by which we fix the proportions of figures; though it much excels both in universality and exactness, the loose judgments of the senses and imagination; yet never attains a perfect precision and exactness. It's first principles are still drawn from the general appearance of the objects; and that appearance can never afford us any security, when we examine, the prodigious minuteness of which nature is susceptible. Our ideas seem to give a perfect assurance, that no two right lines can have a common segment; but if we consider these ideas, we shall find, that they always suppose a sensible inclination of the two lines, and that where the angle they form is extremely small, we have no standard

> of a straight line so precise as to assure us of the truth of this proposition. It is the same case with most of the primary decisions of the mathematics.
> There remain, therefore, algebra and arithmetic as the only sciences, in which we can carry on a chain of reasoning to any degree of intricacy, and yet preserve a perfect exactness and certainty. We are possest of a precise standard, by which we can judge of the equality and proportion of numbers; and according as they correspond or not to that standard, we determine their relations, without any possibility of error. When two numbers are so combined, as that the one has always a unite answering to every unite of the other, we pronounce them equal; and it is for want of such a standard of equality in extension, that geometry can scarce be esteemed a perfect and infallible science."
> (The word numbers here refer to a collection since the word set was not in usage at the time.)

It is crucial to make a note here of the kind of language that has been used, - '. . . a unite answering to every unite of the other'. This language foreshadows the mathematical concept of a one-to-one correspondence or bijection between sets. The interesting part is that Hume chooses to 'pronounce them as equal'.

If Leibniz had made similar deliberations in his mind, it would put him in an uneasy situation because it would violate his seminal principle called Indiscernibility of Identicals. Leibniz put forward this ontological principle (along with the Identity of Indiscernibles) which became so widely used and agreed upon that it later came to be known as Leibniz' law. Simply put, it states that if entity A is identical to B, then A and B have the same properties. Leibniz could see quite easily that one could put the natural numbers and the set of even numbers in a one-to-one correspondence.

$1 \rightarrow 2$

$2 \rightarrow 4$

$3 \rightarrow 6$

$4 \rightarrow 8$

And so on. . .

But to say that the set of natural numbers and the 'seemingly smaller' set of even numbers had exactly the same properties was an idea Leibniz was not able to accept. This is a plausible reason why in spite of his conviction on the existence of the 'actual infinite' he could not accept the position Cantor would take.

(It would be negligent at this juncture if we don't admit that Hume's writing and Leibniz' deliberations were a few decades apart but we can convincingly argue that the essence of Hume's writings was completely contained in the deliberations on the Galilean paradox and Euclid's fifth axiom (part-whole axiom) and Leibniz had written on these topics at length. Leibniz had indicated that he was reluctant to break away from the tradition set by Euclid's fifth which states that the whole is greater than the part; the reasons for that reluctance can be exactly the one we have outlined above.)

We shall revisit Leibniz and his views on the infinite and how they helped shape his theory of the unconscious via his law of continuity, but for now we shall turn our attention to Cantor.

In spite of his respect for Leibniz' metaphysics, Cantor, unlike Leibniz, did not feel the need to be restricted by them. His idea of infinity was remarkably different from what anyone could conceive of at the time and the cultural resistance to his set theory was inglorious and oppressing. The truly radical idea in his theory was the rejection of the Euclid's fifth axiom (the whole is greater than the part). It stems from the idea that since that comparison was only meaningful while comparing finite sets of objects, it is at best a contingent truth and can easily not be valid for infinite sets. Instead of finding ways to subvert Galileo's paradox, Cantor embraces the spirit of the paradox.

(Galileo's paradox could be restated as: There is a one to one correspondence between each number and its squares.

$1 \rightarrow 1$

$2 \rightarrow 4$

$3 \rightarrow 9$

. . . . . .

And so on. Which convinced Galileo that since there is for every square there is exactly one positive square root so it would seem one set could not be greater than the other. He concluded that the words less, greater and equal loses significance in a discussion of infinite sets.)

Instead of being uneasy about the nature of this problem, Cantor chooses to make this the bedrock of his theory. He follows Dedekind's definition that an infinite set is precisely one which has a one to one correspondence with a proper subset. When it came to infinite sets he had two laws. The first was that if there existed a bijection (one-to one correspondence) between them they were equivalent. He ascribed to them the same cardinality. The second was that the set of integers had an infinite number of elements whose cardinality he denoted by 0. The cardinality of power sets of every infinite set was of a greater cardinality than the corresponding sets which initiated the study of cardinal numbers and the transinfinite. We have thus successfully demonstrated how Cantor applied cardinality to the construction of a mathematical actuality, thus completing his break from Leibnizian infinity while always paying homage to it.

Cantor's theory had gained almost universal acceptability in most mathematical communities (although many alternative formalisms exist). However, there were a few mathematical questions that needed to be resolved like Russell's paradox and so on which led to refinements like Russell's Type theory and to the much used ZFC (Zermelo-Fraenkel axioms with the axiom of choice).

At this juncture we should also mention the formulation of the Schroder- Bernstein theorem which states that given two infinite sets A and B such that there is an injective (one-to-one mapping) from A to B and also an injective mapping from B to A then there also exists a bijective map from A to B. Or in Cantor's formulation they belong to the same class of infinity. This theorem gives us an extremely important classification enabling us to compare and in some way equate between the cardinalities of different infinite sets. We should also mention that in spite of the naming of the theorem, the result was discovered by Cantor and the first proof available was given by Dedekind.

From a philosophical standpoint this theorem is significant because it shows us that even in Cantor's formalism of 'actual infinity' the construction is not absolute and with his formalism we can compare and classify different kinds of infinite sets. But even Cantor realises that the iterations of subsequent alephs and their formulation of being the power set of one after the other creates a different paradigm of infinity which cannot be captured in the paradigm he creates.

In Cantor's own words to a letter which he wrote to Richard Dedekind he says, "I have never proceeded from any Genus supremum of the actual infinite. Quite the contrary, I have rigorously proved that there is absolutely no Genus supremum of the actual infinite. What surpasses all that is finite and transfinite is no Genus; it is the single, completely individual unity in which everything is included, which includes the Absolute, incomprehensible to the human understanding. This is the Actus Purissimus, which by many is called God."

Infinity in the Indian context

If we turn our attention to Indian context, we will see that interesting developments had been taking place for centuries which require closer scrutiny.

In his book *The Crest of the Peacock: Non-European Roots of Mathematics*, George Joseph notes how as early in the sixth century BCE, Jain mathematicians had delineated the infinite into five categories: infinite in one direction, infinite in two directions, infinite in area, infinite everywhere and perpetually infinite. While from the categorization it is clear that their understanding of infinity stemmed from a spatio-temporal paradigm, we must also recognize the fact that they were also willing to transcend that through their conception of 'infinite everywhere'. This conception being quite close to the idea of an actual infinity which we have discussed previously. Now, we have seen how Leibniz's rejection of the Aristotelian orthodoxy regarding infinity led him to cement his theory of calculus. Let us stop at this juncture to analyse the work of Madhava, the founder of the Kerala school of Astronomy and Mathematics. He had formulated the infinite series representations of trigonometric functions which would only appear in western mathematics a couple of centuries later, using formal methods in calculus like Taylor's expansion and so on. This leads a branch of historians to point out claims of eurocentrism in outlook towards modern mathematics and make the claim that early formulations of calculus were thus made in India much before the west. Instead of choosing to invest in that line of questioning we take a far more interesting look at the significance of what Madhava had achieved. His result shows us that it is possible to conceive of basic calculus with an Aristotelian framework. It is not necessary for us to adopt Leibniz philosophical take on infinity in order to establish basic calculus. In fact, Madhava and his successors would go on to find infinite series, expansions for Pi as well based on their knowledge and treatment of infinite continued fractions. In fact the question of the influence of infinite continued fractions in the development of mathematical ideas in the far south is an interesting question that should also be taken up in greater detail as we can see that influence persist through the centuries to influence even Ramanujan who made remarkable contributions in this field of enquiry and completely challenged our preconceived notions of infinity.